SELECTED PIAGETIAN TASKS AND THE ACQUISITION OF THE FRACTION CONCEPT IN REMEDIAL STUDENTS

BY

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This clinical study was designed to answer two questions: 1, is there a relationship between the acquisition of cognitive structures, as exemplified in Piaget-type tasks, and the acquisition of the fraction concept; and 2, is there any difference between the concrete or manipulable and the pictorial or written modes of presentation in assessing students' knowledge?

Three instruments were developed. The first was a set of tasks, similar to those used by Piaget to test for the cognitive structures thought to be related to the concept of fraction: conservation of number, seriation, classification, class inclusion, conservation of distance, and conservation of area. Tasks were prepared in concrete or manipulable and pictorial forms.

The other two instruments were fractions tests, one concrete or manipulable, and one written, containing

parallel sections on the concept of fraction (discrete, number line, and area models) and equivalence and comparison of fractions. The written test also included addition and subtraction of fractions with like denominators.

A pilot study was conducted with four students at Gainesville High School, Gainesville, Florida, in summer, 1979. The main study was done in spring, 1980, with 10 girls and 15 boys in the tenth, eleventh, and twelfth grades (median age 16 years), who were enrolled in the compensatory mathematics classes in Eastside High School, Gainesville, Florida.

Tests were administered individually; interviews were recorded. The tasks were administered first. The two fractions tests were given on the next available day, 12 students taking the concrete form first and 13 taking the written test first.

In general, students scored very low. No students were successful on conservation of area tasks; 8% were successful on classification tasks. The best scores were 56%, conservation of distance; 44%, seriation; and 36%, conservation of number.

No student passed all sections of either fractions test. Three students passed both forms on concept of fractions, discrete model. On the concrete form, scores were better on the discrete and area models of the concept of fraction (39% and 56%, respectively, being the average percentage of students correct per item in those sections) than on the number line model (average of 14% correct per item). Performance was poor on the equivalent fractions section (average of 19% correct per item); no student could do the comparison of fractions task.

On the written test, results were similar except on equivalent fractions: 2 students (8%) passed the section, and 7 other students (28%) answered 3 out of 4 items correctly, apparently using a reducing algorithm. Six students could add and subtract fractions, but were incorrect on many items related to the concept of fraction.

To answer the two main questions, data were examined using Walbesser contingency tables. No strong trends were evident, but there were some patterns. Students who could conserve number performed more satisfactorily on the discrete model of fraction. Students performed better on the concrete form of the class inclusion task than on the pictorial (56 to 8%).

Students taking the concrete form of the fractions test first were more successful on the written test than those who took the written test first, indicating that learning may have occurred during the administration of the concrete test.

CHAPTER ONE
INTRODUCTION

Students who enter secondary school have all received some instruction in basic mathematical concepts and skills. In spite of having received instruction, there exists a group of secondary school students who apparently have not learned these concepts and skills.

The majority of students are able to master operations with whole numbers, except possibly long division. The introduction of fractions signals the beginning of a dramatic separation of students into those who succeed in mathematics and those who do not. Many frustrated educators have seen the advent of the hand-held calculator as salvation for those students who cannot operate with fractions; they say that students will no longer be blocked from further progress in mathematics because they can not remember when to "invert," etc. Others feel that real progress will still be missing unless students have a basic understanding of the concept of fraction, on which the concepts of decimals and percentages are logically based.

Those responsible for educating the masses realize that there is a lower end to every distribution; a normed test will always have Stanines 1 and 2. But in recent years, there has been an attempt in many states to agree upon specific mathematics competencies for all students. These competencies would become the minimum achievement required for graduation from high school. This idea represents a considerable advance over the old "grading on the curve" method, in which the person who did the "worst" automatically failed. Competency testing also has its imperfections. Nevertheless, only after competencies were chosen and tests administered did educators become aware of the large number of students who had not mastered these minimum skills. The finding was consistent with the results of the 1972-73 mathematics assessment by the National Assessment of Educational Progress (NAEP).

In many secondary and adult schools, compensatory or remedial instruction is scheduled for students who have not mastered basic skills. Often smaller classes are arranged; teaching aides or assistants are sometimes provided to lower the student-teacher ratio even more. However, the instruction appears to consist of essentially the same strategies that did not produce competence in previous years of schooling. Prejudicing chances for success, some schools assign their beginning teachers to these compensatory classes, the more experienced teachers being given the brighter students and the college preparatory subjects. Compounding the dilemma still further is the current shortage of mathematics teachers; teachers teaching "out of field" are often assigned to compensatory classes.

It seems evident that it is not a simple task to compensate for the learning not yet achieved in eight, nine, or ten years of school; it will require experience and knowledge both of the subject matter to be taught and of how students learn. It will probably not be accomplished by repetitious drill or by more of the same instruction that was not successful in the past.

Fifteen years of teaching in various school settings, producing some success with these challenging students, have given this investigator a basic belief, which is as follows: To remediate or compensate mathematical deficiencies, first, one must start where the student is; secondly, the student usually needs a hands-on or manipulative approach to discover concepts for himself or herself. This study provides an opportunity to begin to see whether these ideas can be substantiated.

One purpose of this exploratory study is to investigate why certain secondary school students have not mastered a specific portion of mathematics content, the fraction concept. What can be learned about students' understanding of fractions could eventually lead to more successful instructional strategies; first, it is necessary to identify what they know. Knowledge of where the individual student is with respect to a mathematical concept has not been gained from standardized tests. More sensitive, individual testing, such as that used in clinical studies, is needed to reveal the student's thinking about mathematics.

The secondary school student has through the years accumulated some information and misinformation about operating with fractions. Some mathematics educators attempt to isolate and identify the errors, and then remediate them. But such a method may not be successful if the student does not have a firm understanding of the concept of fraction on which to base the operations. What is the necessary foundation?

Two things are needed: better diagnosis of what a student knows now about mathematics, and knowledge of what basic structures are required for the student to be able to learn mathematics. In trying to meet the first need, better diagnosis, one is led to the methods of Jean Piaget, who pioneered in the use of the clinical interview to try to understand children's thinking. His results in turn lead to possible insights into the second need. For in Piaget's work with young children, there are described behaviors remarkably like those observed in secondary school students who were having trouble with fractions. For some reason the cognitive development of certain students has been delayed, so that they do not have the cognitive structures often assumed to be present in all secondary school students. The question arises: is this a coincidence, or are the two deficiencies related? Could their tardy cognitive development be the cause of some students' difficulties in learning mathematics? It is not surprising to find that students who are behind in one academic area are slow in something else as well. But if specific Piagetian concepts are found to be related to the specific mathematical concepts to be taught, the finding could be very helpful to those who diagnose student deficiencies and plan instruction. In this study an attempt is made to identify basic cognitive structures necessary for an understanding of the fraction concept.

Another interest of some mathematics educators, especially those involved in elementary school mathematics, is the laboratory method of teaching mathematics. The method is not new; the resurgent interest can probably be traced to the late sixties and the Nuffield Project in England. The use of concrete or manipulable objects in the learning of mathematics is fairly well established in elementary schools, but is rarely seen in secondary schools. A second purpose of this study is to consider whether there may be any justification for the use of these materials in secondary schools.

In a specific portion of mathematics content, the concept of fraction, this study will explore the following two questions:

Question 1. Is a particular level of cognitive development, as indicated by performance on certain Piaget-type tasks, prerequisite to the student's acquisition of the fraction concept?

Question 2. Does the mode of presentation of mathematics content, concrete or symbolic, make a difference in students' performances? Or are students who are successful at a task in one format always able to perform the same task when it is presented in the other manner?

CHAPTER TWO
RESEARCH REVIEW AND RATIONALE

In designing instruction on fractions, an educator must consider the goal, or what the student is to learn about fractions, and the present status, or what the student already knows about fractions. The teacher's instructional plan is the strategy that is expected to effect movement from the present status toward the goal.

Mathematics educators have studied the instructional process from both ends. In this section, the goal, the mathematics content to be taught, will be considered first. Pertinent research about the student's acquisition of the fraction concept will be reviewed. Next, attention will be given to the learner's competence with respect to the goal. The congitive development theory of Jean Piaget will be presented as the theoretical framework for describing the development and knowledge of the learner with respect to fractions.

The third section will consider the interaction between the learner and the mathematics content. Related aspects to be discussed include Piaget's study of how children understand fractions and educators' attempts to assess the learners' present knowledge of fractions. Current research trends to be reviewed include diagnostic and prescriptive studies; studies of students disadvantaged in mathematics;

studies employing a clinical method; and studies which attempt to apply Piagetian theory to education. The ways in which students respond to the concrete and symbolic modes of presentation of mathematics content will also be examined.

The Content: Fractions

In considering mathematics content, not everyone agrees that the understanding of, and computation with, fractions are worthy goals. In an article on the metric system and mathematics curriculum, for example, Sawada and Sigurdson (1976) suggest that common fractions should be studied only at the conceptual level, and that decimal numeration should receive major attention.

Fraction Hierarchies

Those researchers who do select fractions as mathematics content to be taught often perform task analyses after the work of Gagne (Gagne, Mayor, Garstens, and Paradise, 1962). A specific concept or skill is analyzed by its subconcepts or subskills. A learning hierarchy, or a network of partially ordered subconcepts or subskills, is developed on the basis of logical relationships (Gagne, Note 1). The assumption is that if the hierarchy is valid, it gives a sequence, perhaps optimal, for teaching the component parts of the concept of skill. However, it appears that these "expert" generated learning hierarchies are not equivalent to student generated learning hierarchies with the same terminal behavior (Walbesser and Eisenberg, 1972). A task analysis technology, described by Resnick (1976) and Walbesser and Eisenberg (1972), has been developed to test the validity of a hierarchy with students, to find out whether the various hypothesized dependencies of the hierarchy are supported.

Greeno (1976) is concerned with showing how psychological theories might be used in formulating instructional objectives. He has attempted to identify the "cognitive objectives" needed to produce the desired outcome behaviors. Presenting his work as a serious proposal about what the goals of instruction are, he says,

> It may be that when we see what kinds of cognitive structures are needed to perform criterion tasks, we will conclude that something important is missing; but if that is the case, it also will be important to identify a more adequate set of criterion tasks in order to ensure that instruction is promoting the structures we think are important. (p. 124)

Adding fractions was Greeno's first example. He constructed a "procedural representation," or a flow chart, for adding fractions. Recognizing that finding equivalent fractions is necessary both before and after the actual addition, he looked at three different models, or procedures, for finding equivalent fractions. The first is based on "spatial processing" of a region (or an area model). The second is a "set-theoretic" (or discrete) model. And the third is simply an algorithm, "operating directly on numerical representations" (p. 133). Had he extended his reasoning one step further, he might have wondered what understanding of area a child would need to be able to use the spatial processing model, or what concept of number, to use the discrete or algorithmic model.

Uprichard and Phillips sought to generate, then validate, a hypothesized hierarchy for adding (1977) and subtracting fractions (Note 2). The authors intended to give consideration "to both psychological and content (discipline) factors" in identifying and hierarchically ordering tasks. The procedures for both studies were essentially the same. Fraction addition and subtraction problems were divided into two levels, those with like and unlike denominators. Within each level, classes were identified by the nature of the denominators, prime or composite, and the nature of the relationship between the two denominators. Further, there were sum or difference categories, depending on various renamings required. Both studies were done with students in grades four through eight; the majority were in the fifth and sixth grades. Items were compared by two methods, the Walbesser (Walbesser & Eisenberg, 1972) contingency table, with ratio levels of acceptability as determined by Phillips (1972); and pattern analysis, after Rimoldi and Grib (1960). The end results were two lists of problems, in order of ascending difficulty. Conclusions were that problems of certain types should be taught before problems of other types, based on the assumption that those missed most often, and therefore, by definition, the hardest must depnd on the easier problems as prerequisites.

Examination of these dependencies yields examples of concepts which could, logically, seem prerequisite to others, but which have a reversed order of difficulty for students. The following example is taken from the subtraction study: It was found that tasks involving whole number sums greater than one (such as 5 - 1/6) were more difficult than those involving mixed numbers (such as 1 1/6 - 2/6) (Uprichard and Phillips, Note 2, p. 10). Perhaps students were performing a rote algorithm on 1 1/6 (denominator times whole number plus numerator), rather than realizing that 1 can be renamed as 6/6 and 5 as 30/6.

The authors say in summary that the results

> support the notion that both epistemological and psychological factors be considered when developing teaching sequences in mathematics. Some of the implications above would not necessarily be derived from logical analysis alone. Also, in interpreting the results of this study one must be conscientious of the limitations of indirect validation procedures. For example, confounding variables such as prior educational experience of subjects and errors of measurement must be considered. (p. 11)

Students older than their subjects will have had even more experiences in school, and the partial learnings they bring to a task may function as a confounding variable. The rote application of a poorly understood, or poorly remembered, algorithm is an example of this. In discussing the Uprichard and Phillips work, Underhill (Note 3) remarked that a hierarchy may be valid for original learning, but not necessarily for remediation when instruction has already been given. There may be some "subskill retention hierarchies" that could be omitted in remediation.

As pointed out by Kieren in his review (1979) of the addition study, Uprichard and Phillips's analysis treated fractions as symbols to be manipulated according to formal algorithms (a very limited view). Kieren further suggested that such studies needed to have a sound epistemological basis from which to work, and that clinical evidence needed to be given to support the statistical analyses. It is known which problems were missed, but it is not know why they were missed.

Novillis (1976) studied more basic subconcepts of the fraction concept, with subjects in grades four, five and six. Each subconcept was depicted by a model that had as its unit either a geometric region (the part-whole model); a set (the part-group model); or a unit segment of a number line (which the author considered a specific form of the part-whole model). The investigator constructed a hierarchy of dependent subconcepts of the fraction concept and designed a fraction concept paper-and-pencil test of 16 subtests, one for each of the subconcepts. Most of the subtests contained one item each of the following types:

a) given a fraction, the student was asked to choose the correct model.

b) given a model, the student was asked to choose the correct fraction.

c) given a model, the student was asked to select another model for the same fraction.

d) given four models, the student was asked to choose the one that did not depict the same fraction as the others.

To validate the hierarchy, Novillis (1976) used a category system equivalent to Walbesser's contingency table, and analyzed the individual dependency relationships using ratios developed by Gagne et al. (1962) and Walbesser (Note 4). Support was found for 18 of the 23 dependencies in the hierarchy.

The author concluded that certain subconcepts were prerequisites to others. The main dependencies are given below, with the subconcept on the left being prerequisite to the subconcept on the right.

Lower order subconcept	Higher order subconcept
associating fractions with part-whole and part-group models	associating a fraction with a point on a number line
associating a fraction with a part-whole model or with a part-group model	using a fraction in a comparison situation involving the respective model
associating a fraction with a part-whole model or with a part-group model	associating a fraction with the respective model where the number of parts was a multiple of the denominator and the parts were arranged in an array that suggested the denominator

associating a fraction with a part-whole model or with a part-group model having congruent parts	associating a fraction with the respective model having noncongruent parts, where (in the case of part-whole models), the parts were equal in area. (Novillis, 1976, p. 143)

The author noted that the study was exploratory but inferred that elementary school students were not exposed to a sufficient variety of instances of the fraction concept or negative instances (cases where it is not valid) to permit generalization of the concept.

Because they are relevant to the present study, two of Novillis's examples are given here:

> Many students can associate the fraction 1/5 with a set of five objects, one of which is shaded, but most cannot associate the fraction 1/5 with a set of ten objects, two of which are shaded, even when the objects are arranged in an array that clearly indicates that one out of every five is shaded. . . .
>
> If two rectangular regions have been separated into five parts such that in one case the parts are congruent and in the other case the parts are neither congruent nor equal in areas, and in each case one of the parts is shaded, then many students associate the fraction 1/5 with each of these regions and indicate that 1/5 of each region is shaded. (p. 143)

Since the instrument for the validation was a written test, the intriguing question of why they missed the items cannot be answered. In the case of the second example, was the difficulty due to their concept of area? A clinical study, in which individual students could have been observed and interviewed, might have yielded further information.

In a discussion of directions for research, Lesh (1975a) suggested that mathematics educators should apply Piagetian techniques and theory to rational numbers, and referred to Kieren's (1975) paper as "a first step in the direction of a Piagetian analysis of the concept of rational numbers" (Lesh, 1975a, p. 15). The work by Kieren seems to be motivated by curriculum development more than by the theory of Piaget. However, of available published work, it is closest in focus to the present study. Therefore it will be discussed at length.

Interpretations of Rational Numbers

Kieren was concerned about the different possible interpretations of fractions, particularly the "algebraic" aspects of operations on rational numbers, which are usually not presented when fractions are introduced, and which sometimes get lost. He attempted to show the connection between the mathematical, cognitive and instructional foundations of rational numbers in the following way: He named seven different interpretations of rational numbers. For each of these interpretations, he stated the mathematical structures emphasized. Then he listed a set of related cognitive structures and a set of instructional structures (or sequences of necessary experiences). It is not always clear whether he was summarizing existing educational practices or whether he was making recommendations for instructional sequences.

Kieren suggested these seven interpretations of rational numbers:

1. Rational numbers as fractions. This is his label for the most common interpretation of rational numbers, the symbols used in computation. In this interpretation, the associated mathematical structure is a set of procedures (or algorithms) for manipulating the symbols. Kieren gave very little attention to the other two kinds of structure for this interpretation:

> The corresponding cognitive structure is a set of skills. It is not necessary under this interpretation to assume any other structures underlying the skills. The prerequisites for these skills would be skills in computation with whole numbers and not developed concepts of part-whole relationships or proportionality.
>
> The major instructional strategy is diagnosis and remediation both based on elaborate task analysis. (Kieren, 1975, p. 107)

It seems doubtful that the student could in fact organize and memorize these skills (the 160 different addition types he mentioned, for example) if there not some other cognitive structures on which to anchor the skills.

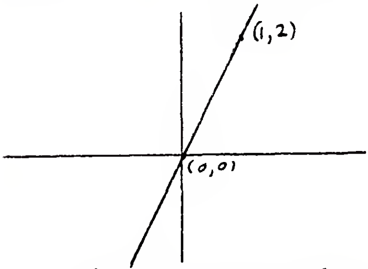
Concerning the instructional strategy, even though Kieren did say the interpretations were not independent (p. 103), this passage might still lead one to believe that this narrow, symbolic interpretation is to be readily found in classrooms. Actually it is highly unlikely that a teacher would present the algorithms for the first time without some attempt to give meaning to the processes by appealing to one or more of the other interpretations, or to some concrete device.

2. Rational numbers as equivalence classes of fractions. A rational number is defined as a set of ordered pairs of integers. In mathematical structure, the rational numbers, together with the operations of addition and multiplication, constitute an ordered field.

The principle underlying concept needed, according to Kieren, is that of an ordered pair of numbers. He sees three phases: perceiving a real situation and its coordinate parts in order, being able to represent these coordinates symbolically, and associating the symbols again with a coordinate reality. Kieren continues:

> With rational numbers the child must learn to identify part-whole situations, learn verbal and numerical codes for these, and learn to correctly identify a code (fraction) with a part-whole setting. As a cognitive capstone of this ordered pair concept set, the child must realize that a part-whole setting can be seen in a set of equivalent ways, and that the various fractions which represent the elements of this set are equivalent. (Kieren, 1975, p. 109)

But logically, it is not necessary to understand anything about part-whole situations to use the equivalence class concept of rational numbers. For example, a rational number (a, b) can be said to be equivalent to (1, 2) if (a - 1) / (b - 2) = 1/2. Alternately, one could give a geometric meaning of equivalence. Referring to Kieren's (1975) graph (p. 108), partially reproduced to



left, one could say that the rational number (a, b) is equivalent to (1, 2) if (a,b) lies on a line through (0, 0) and (1, 2).

Kieren suggests that the proper instructional strategy for this conceptual development is exposure to a wide variety of part-whole settings. He mentions four settings: state-state (static comparison between a set and one of its subsets), state-operator (divide 3 cookies among 5 persons), operator-state (use 5 of a dozen eggs) and operator-operator (cut a pie in eighths, serve 5). The student must also understand that these ordered pairs are numbers. This understanding must include the relationship of this new set of numbers to whole numbers, and a "notion of operations consistent with the fractional and equivalence notions" (p. 110). This second notion, Kieren feels, depends on the ability to partition both discrete and continuous quantities. Examples he gives are: dividing 15 plants among 5 pots, dividing a rope into 5 equal pieces, and dividing some crackers among 4 people.

In concluding this section, he says that in this interpretation,

> the child must be able to assign a pair of numbers to a part-whole situation. This, of course, entails the ability to logically handle the part-whole relationship in both the discrete and continuous cases. The ability to handle class inclusion may be very important in the former case, while partitioning plays a role in the latter. (Kieren, 1975, p. 110)

3. Rational numbers as ratio numbers. An example of a ratio number is the number x, where x is to 1 as 1 is to 8. This interpretation leans heavily on the previous one, as it depends on ordered pairs and operations proceeding from equivalence classes. This interpretation is a sophisticated one and Kieren does not expect the child to be able to deal with it until the proportionality schema is developed, probably not until later adolescence.

4. Rational numbers as operators or mappings. In this interpretation, 2/3 is an operator which maps 3 onto 2, yielding a line segment 2/3 as long as the original. A finite analog would be giving 2 boxes of crayons to every 3 children. (Thus 6 children would need 4 boxes, etc.; equivalence can be seen in operators in this way.) Of the operations, multiplication and division can each be thought of as one operator following another, and are easier than addition and subtraction. Kieren says that three cognitive structures are critical to this interpretation. One is the notion of proportion. However, he says,

> the rational number notions in this interpretation can be developed as concrete generalizations about a large number of concrete situations. Thus, these notions from the point of view of the child can be considered preproportional. It should also be noted that the fraction notion in this interpretation is based on the quantitative comparison of two sets or two objects; hence, part-whole or class inclusion notions are not central to the interpretation. (Kieren, 1975, p. 115)

The child must also have a structure of composition (one operator followed by another) and be able to replace these transformations by their product.

The third structure is that of properties, particularly those of inverse and identity, and the underlying reversibility notion.

Instructional strategies would include work with similar figures, which Kieren calls "preproportional," and exchange games with finite sets.

5. Rational numbers as elements of a quotient field. The rational number x is a solution to an equation of the form ax = b, where a and b are integers. Field axioms are assumed. This interpretation relates rationals to abstract algebraic systems, and "is not closely related to the natural thought of the child" (Kieren, 1975, p. 121). Because it requires formal reasoning, this interpretation will not be detailed further.

However, Kieren says that the more primitive cognitive structure underlying the quotient concept is partitioning: if there are 6 pizzas for 5 children, what is an equal share for each? His simpler illustration is this: Here are 20 letters to be divided evenly in 5 mailboxes. This problem can be solved by distribution of the letters one at a time into the mailboxes, like dealing out cards (Kieren, 1975, p. 121).

6. Rational numbers as measures. Rational numbers are points on the number line. Addition is the simple laying of two vectors end-to-end and reading the result. This interpretation gives an intuitive notion of order.

Kieren gives the cognitive structures that seem particularly important:

> The first is the notion of a unit and its arbitrary division. The child must realize that one can partition the unit into any number of congruent parts. Second, the child must be able to conceptualize part-whole relationships in this context and recognize equivalent settings arising from partitioning of the unit (1/2 = 3/6). Third, the child must develop the concept of an order relation. This involves both the ability to order physical reality and the ability to use correctly the symbolic order statements. Underlying these structures are more general structures, conservation of length and substance, and a general notion of ordinal number. (p. 125)

Instructional activities are suggested by both forms of division, measurement and partitioning. Equivalences can be shown with rods or paper strips of different colors.

7. Rational numbers as decimal fractions. In this interpretation, rational numbers are those which can be expressed as either terminating or repeating decimals.

The operations are extended from those for whole numbers, making computation simple. In division, a remainder is not needed. Teaching from this viewpoint would not provide pre-experience for the rational expressions of algebra.

The cognitive structures necessary are similar to those for measurement. However, the child must be able "to generalize in the symbolic domain" (Kieren, 1975, p. 126). Also, one out of six parts, or 1/6, is a natural extension of counting; saying "about .16" is not. Therefore measuring and estimating are critical. Estimating, he says, involves a general notion of unit and the ability to think hypothetically.

Instructional activities would include any work with the numeration system, and operations with whole numbers. Metric system measurement and money also provide natural models for decimal fractions. And estimating length is a pre-decimal fraction activity. Kieren says that "the processes of seriating and comparing are of paramount importance as is the whole notion of order" (p. 127).

Having described these interpretations of fractions, Kieren makes the point that all should be considered in the curriculum. Given these interpretations, he says, a curriculum developer-instructional designer "can then ascertain the necessary cognitive structures for meeting the objectives and develop sequences of instructional activities which contribute to the growth of these structures" (Kieren, 1975, p. 128). He says further that

> a researcher who asks, "How does the child know rational numbers?" must go through a similar process. He can study selected interpretations in more detail and identify what he believes to be the most important cognitive structures. Settings can then be developed or used which allow one to see the extent to which a child has such structures. The growth of such structures can then be studied developmentally. Alternatively, the importance of such structures can be tested. Here, one would test the effect of having or not having some structure on attaining some rational number objectives. (p. 128)

As mentioned elsewhere, students' learning does not always proceed logically, or according to researchers' expectations. Therefore, to ascertain these necessary cognitive structures as Kieren suggests may require in-depth study of students and their learning.

Kieren then summarizes the "conglomerate picture of rationals," including some work that has been done in developing hierarchies of skills, and suggests curriculum research. He further suggests clinical research such as that of Inhelder and Piaget (1969) on the growth of logical thinking, saying,

> Some aspects and behaviors of rational number will be impossible to study in their "natural state." They will undoubtedly be colored by instructional experience. (p. 140)

As already observed, secondary students will have had many such instructional experiences, which may confound the study of their concept of fraction.

Kieren also says that "it would seem that conservation of area and length might be related to continuous partitive division" (Kieren, 1975, p. 141). This idea will be discussed further in following sections.

The Learner

Two aspects of the learner will be considered: first, what is known about the learner's cognitive structure, as described in the cognitive development theory of Piaget; and secondly, what is known about the remedial student.

Not all scholars agree with everything Piaget says. In fact, according to Flavell (1963),

> the system has an extraordinary penchant for eliciting critical reactions in whoever reads it. Piaget has done and said so much in a busy lifetime that foci for possible contention and disagreement abound. More than that, he has consistently done and said things that run so counter to accepted practice as to make for an immediate critical reaction in his reader, almost as though he had deliberately set out to provoke it. (p. 405)

Flavell also disagrees with certain parts of the theory, but concludes that Piaget's work "is of considerable value and importance, with a very great deal to contribute to present understanding and future study in the are of human development" (p. 405).

Piaget's theory of cognitive development is not a theory of education, but of knowing, which may or may not be related to the knowledge purveyed in schools. Piaget has left educational implications and appplications to others (Sigel, 1978, p. xvii). However, the following discussion of his theory will indicate that the cognitive structures he describes are of importance to school learning.

Piaget's Theory

Piaget is an epistemologist. He studies the nature of knowledge; he is concerned with finding out how the ability to know develops. He looks for commonalities in children's knowing that do not depend on what school they attend, their emotional state or other factors (important though they may be to the overall condition of the child).

During decades of study on hundreds of children, Piaget concluded that there were definite levels of cognitive development which were invariant in the sequence in which they ememrged. Unfortunately, this idea gives rise to the first of many common misinterpretations of Piaget's ideas. An example is the following:

> The research of Piaget, et al. suggest that all students by about the age of 12, should be able to correctly use an external frame of reference to properly predict water level, pendulum positon, etc. (Dockweiler, 1980, p. 214)

A review of the work cited (Piaget and Inhelder, 1956) fails to turn up the suggestion by Piaget and Inhelder that any student should do anything at a particular age. Piaget does not view cognitive development level as age dependent.

Flavell emphasizes Piaget's position on the stage-age question:

> Piaget readily admits that all manner of variables may affect the chronological age at which a given stage of functioning is dominant in a given child: intelligence, previous experience, the culture in which the child lives, etc. For this reason, he cautions against an overliteral identification of stage with age and asserts that his own findings give rough estimates at best of the mean ages at which various stages are achieved in the cultural milieu from which his subjects are drawn. . . . Of course not all individuals need achieve the final states of development. . . . Piaget has also for a long time freely conceded that not all "normal" adults, even within one culture, end up at a common genetic level; adults show adult thought only in those content areas in which they have been socialized. (Flavell, 1963, p. 20)

The present study is not focused on the chronological age at which a student has reached a stage; all of the subjects are "behind" Piaget's children. Attention is given instead to whether the student has reached a stage, and whether having reached it has anything to do with enabling the acquisition of mathematics concepts.

In this discussion, the major developmental stages themselves will be called "periods," in accordance with Piaget's stated preferences (Flavell, 1963, p. 85); the word "stage" will refer to subdivisions with the period (except where reference is made to authors who use the former nomenclature).

The first period, called the sensorimotor period, lasts from birth to about two years of age. The last one, the formal operational period, in which an individual becomes able to think about thoughts and reason about reasoning, has been found by Piaget to be completed at about age 15. These periods at the two ends of the developmental scale were not exhibited by the students in this study. Therefore, attention will be paid only to the middle period, called by Flavell "the period of preparation for and organization of concrete operations" (Flavell, 1963, p. 86).

The first of two major subperiods is that of preoperational representations, and the second is that of concrete operations.

In the preoperational subperiod, found by Piaget to last roughly from 2 to 7 years of age, the child is learning to use language as representation of thought. His understanding of space increases to include such concepts as more and less, larger and smaller, before and after. He learns to discriminate differences in objects, colors, etc. Yet, in the preoperational child, perception is a stronger influence than reason.

During the concrete operational subperiod, about 7 to 11 years in Piaget's findings, the child acquires a conservation schema. She can classify objects on the basis of a common characteristic. She learns to seriate, or put things in order from smallest to largest or vice versa.

The following are some examples of Piaget-type tasks and how children react in each of the two subperiods.

Conservation of number. In Piaget's theoretical analysis, the concept of number is derived from "a synthesis of class inclusion and seriation" (Sinclair, 1971, p. 152). Piaget's own volume, The Child's Conception of Number (1965), includes conservation of quantity, one-to-one correspondence, logical classification and order relations, each of which were given at least one chapter.

In spite of its complexity, conservation of number has been chosen to present first, because it can make a vivid illustration of what Piaget means by conservation. What will be presented here is a simplistic version, with emphasis on the tasks themselves, based on work by Copeland (1979), Formanek and Gurian (1976), and Lesh (1975b).

A child is shown two rows of beads displayed as follows:

0 0 0 0 0

0 0 0 0 0

The child agrees that there are the same number of beads in each row.

If one row is now spread out, like this,

0 0 0 0 0

0   0   0   0   0

the child of 5 or 6 years may think that there are more beads in the bottom set, because the row is longer. Even if he counts each row, he may still be influenced by what he sees, the length of the rows, in making a judgment about which set contains more beads. He would be said to have the ability to conserve number if, in this case, he could realize that the number of beads remained constant even when the beads were rearranged.

There are three stages into which children's responses can be divided:

I: Says that the second row contains more beads. Pressed for a reason, says, "Because I can tell by looking," etc.

II. Is transitional. Appears to conserve, but is not sure; counts to see whether the rows are equal in number.

III. Realizes that rearranging does not change the number. Asked for a reason, replies, "Because you didn't put any more or take any away."

These three stages are typical of the sequences Piaget finds in other conservation tasks (quantity, length, area, volume, mass). The stages are summarized by Flavell (1963):

> I, no conservation; II, conflict between conservation and nonconservation, with perception and logic alternately getting the upper hand; and III, a stable and logically certain conservation. (pp. 312-313)

Flavell also says that, in this task, "a genuine concept of cardinal number is by no means guaranteed by the ability to mouth appropriate numerical terminology [or count] in the presence of objects" (p. 313).

<u>Seriation</u>. Seriation is the act of putting things in order. Its beginnings are in the child's broad discriminations between big and little.

In one version of the seriation task, a child is given about 10 sticks of different lengths and is asked to put them in order. In stage I, a child can order two, or maybe three, sticks at a time, but there is no overall scheme. In stage III, the child has a plan, and methodically selects the longest (or shortest), then the next longest (or next shortest), etc., and completes the series efficiently. If some sticks are introduced as having been "forgotten," the stage III child can insert them with no problem (Copeland, 1979, p. 96).

The stage II child can usually form the series by "trial and look." as Copeland calls it (p. 96). But a plan or system is noticeably absent. In fact, the child may end up with two or three unconnected subseries, as Lesh (1975b, p. 97) shows:

Even if such a child can complete the series, he may still be unable to insert a "forgotten" stick, Lesh says (p. 97). Copeland says this child "considers the series already built to be complete and feels no need to insert the additional sticks" (Copeland, 1979, p. 94).

In Piaget's language, the preoperatory levels, stages I and II, "lack coordination in that subjects can put two or three elements in order at a time but cannot put all the elements in order. The operatory level sees a general (reversible and transitive) coordination linking these specific actions into a whole" (Piaget, 1976, pp. 300-301).

One source of difficulty in the seriation task lies in the tendency of some stage I and II children to make pairs. It may not be simply that they can only attend to two at a time (that is, they can consider $a < b$, but not $a < b$ and $b < c$ simultaneously). Another factor may be, Piaget says, that

> the conceptualization on which the cognizance is based, which starts from the results of the act, is not only incomplete but often incorrect as well, because the child's preconceived ideas influence his reading of the situation--that is, he sees what he thinks he ought to see. (1976, p. 300)

In this context, once a stage I child picks up two sticks and orders them, she may continue making pairs in that fashion, disregarding the original instructions, because she hears the directions she thinks she ought to hear.

Classification. In a simple classification task, a child is given a collection of objects or pictures and asked to put "the ones that are alike" together. Flavell's (1963) discussion of children's responses will be abbreviated. In stage I,

> the child tends to organize classifiable material, not into a hierarchy of classes and subclasses founded on similarities and differences among

> objects, but into what the authors [Piaget and Inhelder] term "figural collections" [like pictures]. . . . It is a relatively planless, step-by-step affair in which the sorting criterion is constantly shifting as new objects accrue to the collection. . . . Partly in consquence of this inch-by-inch procedure bereft of a general plan, the collection finally achieved is not a logical class at all but a complex figure (hence figural collection). The figure may be a meaningful object, e.g., the child decides (often post hoc) that this aggregation of objects is "a house." Or instead, it may simply be a more or less meaningless configuration. . . . Frequently, at least part of the child's collection is founded on a similarity-of-attribures basis. What often happens is that the child begins by putting similar objects together, as though a genuine classification were in progress, and then "spoils" it by incorporating his "class" into a nonclass, configurational whole. (Flavell, 1963, pp. 304-305)

Flavell says that the stage I child may also begin by putting squares together, but fails to include all the squares or contaminates his collection with nonsquares. This is an illustration of his inability to differentiate, and hence coordinate "class <u>comprehension</u> (the sum of qualities which define membership in a logical class) and class <u>extension</u> (the sum total of objects which possess these criterial qualities)" (p. 305). He explains:

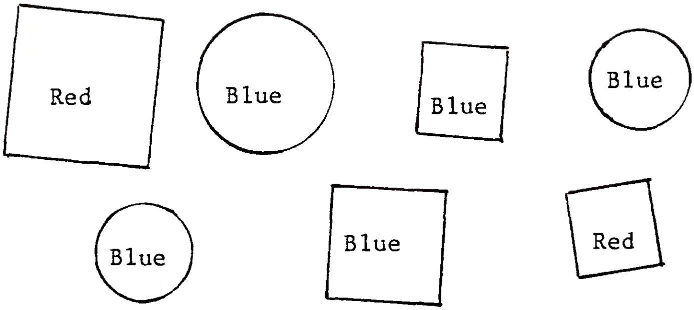
> In a genuine classification, these two properties must always be in strict correspondence: the definition of the classification basis determines precisely which objects must constitute its extension, and the nature of the objects in a given collection places tight constraints on the definition of the class they together form. But for the young child, there seems to be no such strict correspondence. (p. 305)

It is noted by Flavell that these gaps in the child's understanding may be hidden. "The child's ability to bandy about classification-relevant phrases (e.g., 'dogs are animals,' "some of these are red,' etc.) either under ordinary questioning or in spontaneous discourse, is likely to be a most unreliable guide" (Flavell, 1963, p. 306).

A stage II child can form nonfigural collections on the basis of similarity of attributes. He can generally assign every object in the display to one or another group. Still troublesome are groups or collections with only one member, or, worse yet, no members. Copeland reports Inhelder and Piaget's (1969) findings: "The concept of the singular class is not operational until eight or nine years of age, and the empty or null class is not operational until ten to eleven years of age" (Copeland, 1979, p. 69).

Stage III does not occur, for Piaget, until the child has mastered class inclusion. This will be treated below as a separate task.

Class inclusion. The important aspects of class inclusion can be exemplified by these two tasks, taken from Flavell (1963, pp. 307-309). In one, the child is to be tested on the notion of "some" and "all" (a reflection of the understanding of class comprehension and extension, discussed above). A series of objects is shown, such as the following collection:



The questions asked take two forms:

a) Are all the blue ones circles? or

Are all the squares red? etc.

b) Are all the circles blue? or

Are all the red ones squares? etc.

Being able to answer questions like those in a does not guarantee that the child can answer questions like those in b.

In the second experiment, the child is shown a set of flowers with a large subclass of primroses and a few other (various) flowers. It is first established that the child understands that the primroses are flowers. Then questions are asked on the "quantification of inclusion" (Flavell, 1963. p. 308):

1) If I took away all the primroses, would there still be flowers left?

2) If I took away all the flowers, would there still be primroses left?

3) Are there more primroses or more flowers?

Strangely enough, some children can answer questions 1 and 2 correctly and still "fail" question 3. In Piaget's interpretation, if B is the set of flowers and A is the subset of primroses,

> The child can recognize that A and A' comprise B _when he focuses attention on the whole B_ (thus, he can perform B = A + A'), but "loses" B (and the fact that A = B - A') when he isolates A as a comparison term. With B momentarily inaccessible as an object of thought, the child cannot do other than compare A with its complement A'. (Flavell, 1963, p. 309) [Flavell's emphasis]

Conservation of distance. Distance and length are not the same thing. Length is the measure of something which takes up space (one-dimensional) and distance is space (one-dimensional) which can be filled up with something. If movement is involved, the situation is complicated further, according to Piaget, Inhelder and Szemińska (1960):

> Questions about the strips of paper . . . may be asked in terms of "static" length or in terms of distances travelled. The answer is not always the same in both cases and the two languages should not be confused. (p. 106)

The conservation task to be discussed below, adapted from Formanek and Gurian (1976, pp. 32-34) concerns the linear space between two points.

Two small toys, such as cowboys or soldiers, are placed about 50 cm apart. The child is asked if the toys seem to be "close together" or "far apart." (Either one is satisfactory; this establishes a frame of reference.)

Then a low screen, or barrier, is placed about midway between the toys, as if it were a fence separating them. The child is then asked whether they are still as close together or as far apart, depending on the child's first reply. The screen is then replaced, first with a larger screen, high enough to hide the two toys from each other, then with an obviously three-dimensional object, like a block of wood. Each time the child is asked to make a judgment about whether the distance has changed and why.

In stage I, children are thrown off by the partition and no longer seem to be able to consider the total distance between the two toys; they will only look at the distance each toy is from the screen.

A stage II child can consider the total distance, but the distance seems less, because the obstruction is taking up space. For them, distance is empty space.

Children in stage III realize that the obstruction is irrelevant to the distance between the two toys; they state confidently that they are just as far apart because they haven't moved.

Conservation of area. Logically, adding one more dimension would tend to complicate matters. The "farm" task, adapted from the version given by Piaget, Inhelder and Szeminska (1960, pp. 262-273), will illustrate some of the complexities involved in considering area.

The child is shown two rectangular sheets of green cardboard and told that they represent fields of grass. It is established that they are the same size, by putting one on top of the other, if necessary. Then a tiny model of a cow is placed on each field and the child is asked whether both cows have the same amount of grass to eat. Thus, the frame of reference is established. Then the investigator begins to change things. Two identical "barns" are added, one to each field, and again the child is asked whether the cows have the same amount of grass. According to the authors, every child says that they have (Piaget, Inhelder and Szeminska, 1960, p. 263).

A second barn is then introduced into each field, but in a different arrangement: in one, the barn is juxtaposed to the previous one; in the other, the second barn is placed elsewhere in the field, not near the first barn. The child is asked the same question; if it is answered correctly, a third barn is added (in a row in one field, spread out in the other), then a fourth barn, and so on.

The authors found results analogous to the previous ones:

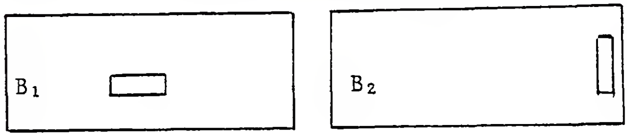
> During stage I we find it difficult to pursue the enquiry, but at stage IIA children are obviously interested, yet they refuse to admit that the remaining areas are equal, often at the very first pair of houses. Here there is no trace of operational composition, and judgment is based entirely on perceptual appearances. At level IIB we find a complete

> range of intermediate responses: up to a certain number of houses the subject admits the remaining meadowlands are equal; beyond that number the perceptual configurations are too different. Here there is intuitive articulation in varying degrees, but not operational composition. At stage III, however, . . . children recognize that the remainders are always equal, relying on an operational handling of the problem which convinces them of the necessity of their reasoning. (Piaget, Inhelder and Szeminska, 1960, pp. 263-264)

Variations in the above procedure produced some surprises. In the discussion above, it is not mentioned where each of the first two barns was placed on each field. In the experiment, they were placed identically. Yet in other experiments, it turned out that if one of the barns was placed in a corner, and the other in the center of the other field, the remaining space did not seem equal to all children (p. 263).

The authors used rectangular "bricks" to represent the barns. They relate one example in which the bricks were first placed in identical positions, and the child identified as GAR agreed that the amount of green was the same. Then;

> [Investigator] And like this (one in the centre of B_1 with the length of the brick parallel with the length of the meadow, another at one end of B_2 and laid breadthways)? [GAR] No, there's more green left here (B_2). [Investigator] Why? [GAR] Because there's all this left (free space). (p. 264)



This situation is reminiscent of some optical illusions in which the orientation changes the appearance of a length.

After carrying out other experiments, the authors noted further that the conservation of "space remaining" did not necessarily occur simultaneously with conservation of "space taken up" (Piaget, Inhelder and Szeminska, 1960, p. 286), and that once an area (or plane surface) had been cut, its area might not seem the same to some children, even when it was put back together (p. 295).

Other features of the theory. There are many other experiments Piaget has done which would be illustrative of his theory and his method. These six tasks were chosen because of their relationship to the present study. Some of the other relevant features of Piaget's theory, taken primarily from Flavell (1963) and Travers (1977), are discussed below.

The child develops a "schema," an organization of ideas or behaviors, a structure in the intellect which enables the child to understand. New information that is found is "assimilated," or added into the existing schema. The process of assimilation entails adding knowledge or behavior consistent with actions already organized within the schema. Later, as the child acts on the environment, the child changes the schema or builds a new one to accommodate new behaviors in response to new situations. "Accommodation" is this building of new schemata or modifying of old schemata to adapt to new situations. A child adapts to the environment by an interplay of assimilation and accommodation.

There are definite stages of cognitive development, invariant in sequence. Each stage is the foundation for the next stage. To go from one stage to the next, the child needs to mature chronologically, and also needs experience with the environment. Further, there has to be a problem that the child wants to solve. The child is not satisfied with the solution produced by the present stage of development; Piaget calls this a state of "disequilibrium." When the child finds a new solution to the problem at a higher cognitive level, equilibrium is restored. This process is called "equilibration." Thus, cognitive development results from the child's interests and drives in interaction with the environment.

Conjectures could be made about what might happen when some of these requirements for development are not met. For example, a child might have matured chronologically without having had the experiences which induce development. The particular environment may not have presented problems that the child wanted to solve. Or the child's interests may have been in art or some other endeavor which did not induce the conflict, or disequilibrium, necessary for cognitive growth. In these cases, the expected structures may not have developed.

Such a situation does not preclude further growth. Piaget's theory does not put a ceiling on development at any age. Therefore, the theory is compatible with the possibility that children of intellectually deprived environments may not yet have achieved the cognitive development of which they are capable. The next section will focus on these disadvantaged students.

Disadvantaged Students

Pikaart and Wilson (1972) "examined the research on the slow learner in mathematics and found it lacking" (p. 41). The meager research that is available, they say, parallels the development of the idea that intelligence is quantifiable. IQ scores are of little use, they say.

> A more fruitful approach . . . is to consider specific learning aptitudes of slow learners and to adapt instruction to take account of these individual differences. (p. 42)

In Suydam's (1971) summary of research on teaching mathematics to disadvantaged pupils, she notes that the summary does not contain many studies done with students in the secondary school. One of the reasons she gives for this is that

> there are not as many slow learners or low achievers or otherwise disadvantaged students still enrolled in mathematics courses in the secondary school. The process of selection or tracking precludes most students in any of the subsets of the disadvantaged from going beyond a general mathematics course. (p. 3)

To "enrolled in mathematics course," she might have added "enrolled in school." With compulsory attendance over at around age 16, many who have not been successful by then drop out.

The studies concerning disadvantaged students that are listed by Suydam (1971) usually focus on comparing different teaching methods, and will be mentioned later. Compensatory and remedial programs have proliferated; still mathematics education researchers interested in secondary school mathematics have devoted the bulk of their resources to studying the students who are in the college bound track, taking courses in algebra and geometry. It is hoped that this study of disadvantaged secondary students will be a start in the direction suggested by Pikaart and Wilson (1972).

The child's development is left now for a consideration of the student's school learning, as the learner interacts with mathematics content.

Interaction of the Learner and the Content

The discussion of what happens when the learner interacts with mathematics content must be limited for this review. The topics chosen can be explained by first summarizing the previous two sections.

First, efforts to study the content, the fraction concept and operations with fractions, were discussed. Included were calls for research to find the underlying cognitive structures of fractions. Secondly, in looking at the learner, relevant aspects of cognitive development theory were described. The lack of research on students who have difficulty learning mathematics was mentioned.

This section, on the interaction of the learner with the fractions content, will relate the preceding sections. For example, in spite of his disavowal of educational objectives in general, Piaget did consider what are almost pre-fraction concepts in some detail. This work on fractions will be reviewed. Next will be a description of assessment efforts aimed at finding out what students in general know about fractions, and then of diagnostic and prescriptive studies, concerned with why the individual student has not learned fractions and what might be done about it. Attention will also be given to clinical studies, which often include detailed observation of interactions between learner and mathematics content. The neo-Piagetian research will be included. Last will be a discussion of the concrete-versus-symbolic modes of presentation of mathematics content in attempts to assess students' knowledge.

Piaget's Fractions

Much of Piaget's work has been done with small children, so he has not given much attention to fractions. He has, however, considered "Subdivision of Areas and the Concept of Fractions" as Chapter 12 of The Child's Conception of Geometry (Piaget, Inhelder and Szeminska, 1960). He describes work with children whose ages range from 4 to around 7 years. He is not studying "fractions" as they are normally taught in school, however. For example, a child is asked to cut a cake, to "divide it up so that the man and the woman will both have the same amount of cake to eat" (Piaget, Inhelder and Szeminska, 1960, p. 304). The child does not have to know either the notation "1/2" or the words "one half" to be able to perform the task. When Piaget writes about "their idea of a fraction" (p. 310), he seems to be talking about the children's idea of partitioning, a basic component of, or perhaps even a pre-concept to, the idea of a fraction. (The fact that the work is a translation may add to the confusion.)

The procedure was as follows:

The child was expected to use a wooden knife to divide a circular cake made of modelling clay equally between two dolls. After the division was performed, the child was asked whether, if the pieces were put back together, it would be equal to the original whole. Those children who could divide the cake into halves were then asked to divide the cake between three dolls, and so on, up to six dolls.

The youngest children often cut two pieces of arbitrary sizes for the two dolls, leaving the remainder of the cake (either ignoring it or pushing it aside). When pressed by the interviewer as to what was to be done with the remainder, a child might refuse to discuss it (p. 305) or even try to hide the leftover part (Piaget, Inhelder, and Szeminska, 1960, p. 306). At this stage the child was concerned neither with equality of shares, nor with exhausting the whole. Some children also seemed to think that two pieces required two cuts. More advanced children could correct their mistake, having made two cuts, by subdividing the remainder and parceling out more cake to the dolls, so that the cake was exhausted, at least, whether equally subdivided or not.

In trying to comprehend the children's behavior, Piaget suggests that the half-to-whole, and generally, part-to-whole, relationship can be understood by the child perceptually. But, he says,

> it is a far cry from such perceptual or sensori-motor part-whole relations to operational subdivision. There are systematic difficulties in understanding part-whole relations on the plane of verbal thinking. . . . When we used phrases like "a part of my bunch of flowers is yellow," or "half of this bunch is yellow," etc., we found that even children of nine or ten thought of the whole bunch as yellow because they thought of the part (or half) as something absolute rather than as being necessarily relative both to the other part (or half) and to the whole. Typical replies were these: "What's a half?--Something you've cut off.--What about the other half?--The other is gone." Obviously the half that is cut off and thought of as a thing apart without reference either to the whole or to the other half echoes the little pieces which are cut off in actual fact by children of two to four. . . . Quite early on children elaborate means of dealing with reality at the level

> of action, and even at the level of concrete operations, but these solutions still need to be re-worked at the level of verbal thinking by means of formal schemata. (Piaget, Inhelder, and Szeminska, 1960, p. 308)

Thus, when a half is cut off, it may become to the child an entity on its own, with no further reference to the whole of which it was a part. Piaget refers to his earlier work on the part-whole relation, when he was studying the child's conception of number:

> Thus, when shown a large number of brown beads alongside two white beads, all these beads being made of wood, the child under seven could not understand that there were more wooden beads than brown beads for he persisted in forgetting about the collection as a whole when concentrating on the brown beads and therefore came to the conclusion: "There are more brown beads than wooden beads because there are only two white beads." (Piaget, Inhelder, and Szeminska, 1960, p. 308)

In the partitioning task, again, subdivision must be reconstructed in thought, but with reference to a concrete situation. It cannot be assumed that a child who is able to physically partition an object can verbalize the actions.

In analyzing the actions of the smallest children, mentioned earlier, Piaget says that the most striking thing is the presence of a part-part, rather than a part-whole, relationship. For them, "the relation between parts is one of juxtaposition and not of a nesting series" (p. 309). The child ignores the quantitative aspect, that two halves are equal, for example, and also the relation of the part to the whole, "from which it may be parted in fact but to which it still relates in thought" (Piaget, Inhelder, and Szeminska, 1960, p. 309).

Piaget's analysis of the fraction concept continues:

> The notion of a fraction depends on two fundamental relations: the relation of part to whole (which is intensive and logical) and the relation of part to part, where the sizes of all other parts of a single whole are compared to that of the first part (a relation which is extensive or metric). (p. 309)

He describes the necessary components of the notion of a fraction as follows:

1. The child must see the whole as composed of separable elements, i.e., divisible. Very young children, he says, see the whole as an inviolable object and refuse to cut it. Later, the children are prepared to cut it, but then the act of cutting it may make the object lose its wholeness.

2. A fraction implies a determinate number of parts. Children who do not realize that the number of shares should correspond to the number of recipients begin by randomly breaking off pieces.

3. The subdivision must be exhaustive, i.e., there must be no remainder. There has been mention already of children who

> refuse to share out the remainder, apparently satisfied that when they have made up the two parts they were asked for, anything left over is neither part nor whole and has nothing to do with the two real parts: these alone are real because these alone go to make up their idea of a fraction. (Piaget, Inhelder, & Szeminska, 1960, p. 310)

4. There is a fixed relationship between the number of subdivisions and the number of intersections, or cuts to be made.

5. The individual parts must be equal.

6. The parts themselves have a dual character: they are parts, but they can also be wholes, and thus are subject to being subdivided further. This is the understanding necessary for finding fourths by halving halves.

7. The sum of the parts equals the original whole. Somehow, cutting the cake changes it for some children. A subject identified as SOM thinks there is more in two half-cakes than there is in one whole (p. 327). Subject GIS says that they are the same, but asked to choose between a whole cake and two halves, chooses the whole, saying, "I get more to eat this way" (p. 320). In a measured understatement, Piaget says, "We see how paradoxical are these replies" (p. 329).

Some of the conditions for understanding the fraction concept may seem obvious, but Piaget has discovered their necessity by seeing their absence in the thinking of children. He further says that these seven conditions must still be part of a general structure to be operational. There must be an anticipatory schema: children must be able to anticipate the solution before they can solve the problem. That is, they must plan ahead where all the cuts will be before they make the first cut. In the absence of such a plan, successive fragmentation of the cake is made. Piaget emphasizes the complexity of the task:

> The subdivision of an area . . . is fraught with considerable difficulty for young children and its complications compare in every respect with those pertaining to logical subdivision of the nesting of partial classes within an inclusive class. (Piaget, Inhelder and Szeminska, 1960, p. 333)

Piaget defines the substages in the subdivision of areas according to whether the seven conditions are met by the children, and whether they can halve, trisect, quarter, etc. (trisecting being more difficult than quartering, since quartering can be done with two successive dichotomies). He concludes the chapter with the following summary:

> The facts studied in this chapter show not merely that there is a clear parallel between the subdivision of continuous areas and that of logical classes, but also that notions of fractions and even of halves depend on a qualitative or intensive substructure. Before parts can be equated in conformity with the extensive characteristics of fractions, they

> must first be constructed as integral parts of a whole which can be de-composed and also re-assembled. Once that notion of part has been constructed it is comparatively easy to equate the several parts. Therefore, while the elaboration of operations of subdivision is a lengthy process, the concept of a fraction follows closely on that of a part. For parts which are subordinated to the whole can also be related to one another, and when this has been achieved, the notion of a fraction is complete. (Piaget, Inhelder and Szeminska, 1960, pp. 334-335)

Although Piaget says that "the notion of a fraction is complete," it must be noted that his discussion has dealt basically with one interpretation of fraction, that of subdividing continuous substances. He studied primarily one medium, clay, which is three-dimensional, though he referred to the task as "subdividing areas." Piaget did try some different shapes and some plane figures, finding it easier, for example, for the children to trisect a rectangle than a circle, and the longer the rectangle, the easier. Some of the children were given a "sausage" of modeling clay and it was found to be the easiest solid to trisect, presumably because it was like an elongated rectangle (p. 319). Piaget's work has given valuable insight into some of the components necessary to a child's concept of fraction. But he did not treat a linear model or a discrete model. He considered only unit fractions, and then with small denominators. He did not look at equivalence, or comparisons between different fractions. And certainly it was not his purpose to study how children learn about fractions as ratios or quotients or how they come to perform mathematical operations with fractions. These other aspects need to be given in-depth study also.

The next topic is students' general knowledge of fractions as taught in school, and as evidenced by assessment tools.

Assessment of Students' Knowledge of Fractions

Students have all received instruction in fractions by the end of the sixth grade, so it is appropriate to ask what understandings and skills they carry with them into junior high and high school.

In view of most elementary school mathematics programs today, Carpenter, Coburn, Reys, and Wilson (1978) say, "13-year-olds should be thoroughly operational with fractions" (p. 34). However, in their summary of the NAEP mathematics assessments, they say that overall results on fraction concept tasks were low. Of all three groups, 13-year-olds, 17-year-olds, and adults, no more than about two thirds responded correctly to an exercise that dealt with fraction concepts (p. 34).

Some of the NAEP results were reviewed earlier (Carpenter, Coburn, Reys, & Wilson, 1976), when only two exercises were released. The first was:

$$1/2 + 1/3 = \quad \text{(p. 137)}$$

Only 42% of 13-year-olds and 66% of 17-year-olds were successful in solving this exercise. Of various incorrect responses, the most common was obtained by adding both the numerators and the denominators (30% and 16%, for 13 and 17-year-olds, respectively). In speculating about this and other errors, the authors say the results suggest "that students are not viewing the fractions as representing quantities but see them as four separate whole numbers to be combined in some fashion or other" (Carpenter, Coburn, Reys, & Wilson, 1976, p. 138).

The multiplication exercise was:

$1/2 \times 1/4 =$ (p. 137)

The students performed better on this exercise, getting 62% and 74% correct answers. The incorrect responses did not show a pattern.

The authors noted that these results were consistent with data from various state assessments and other research (p. 139).

The later, more complete report (Carpenter, Coburn, Reys, & Wilson, 1978) describes the testing of the concept of fraction:

Asked what fractional part of a small set of marbles was blue, 65% of 13-year-olds answered correctly (p. 37).

The three older groups were given this problem:

There are 13 boys and 15 girls in a group.
What fractional part of the group is boys? (p. 38)

Described as "very disappointing," the results were 20%, 36%, and 25% correct answers, for 13-year-olds, 17-year-olds, and adults (Carpenter, Coburn, Reys, & Wilson, 1978, p. 38). The authors commented that

> the cause of the errors cannot be determined from the data. Perhaps there is a problem with the language "fractional part" that would contribute to the "I don't know" responses. But the committed errors must be due to a lack of mastery of fraction concepts and their application to problem contexts. (p. 38)

In a multiple choice exercise, two common fractions less than 1 were given; respondents were asked to select another fraction between them. Correct answers were given by 56% and 83% of 13 and 17-year-olds (p. 39). However, when given six very common fractions less than 1 and asked to write them in order from smallest to largest, "no age group could perform this task adequately" (p. 39).

Asked which fraction was the greatest of 2/3, 3/4, 4/5, and 5/8, 26% of 13-year-olds and 49% of 17-year-olds answered correctly. The authors say,

> The strongest distractor for both the 13-year-olds and the 17-year-olds was 2/3. The exercise clearly shows that 13-year-olds are not yet operational with fractions. (p. 40)

In a survey intended to find out whether deficiencies in fractions skills were due to current instructional programs, Ginther, Ng, and Begle (1976) went to "the most advantaged schools" in their area and tested about 1,500 eighth graders. The students were in intact classes identified as average by their teachers. A battery of fractions tests was designed, to include the cognitive levels of computation, comprehension, and application. In the computation section, the second easiest problem (the second highest percentage correct) is given for each of the operations, with the percentage of students correct:

1/6	3/4	5/8 x 32	7/8 ÷ 5/16
+ 5/8	- 2/5		
63% correct	58%	65%	40%

(Ginther, Ng, & Begle, 1976, pp. 3-4)

The authors were apparently not concerned by these low percentages, commenting in the conclusion that the students had a reasonable understanding of the fraction concept (p. 9). They did, however, decry the students' lack of understanding of structure.

In the comprehension section, items were intended to be answered very easily by students who understood the structure of the rational number system. The following are examples, presumably still the second easiest:

2 x ☐ = 1

42% correct

(p. 4)

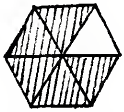
A is

1/2 1/3 1/4 3/8 5/8

62% correct

(p. 5)

The fifth easiest diagram question was as follows;

> There is a drawing on the left. Part of the drawing is shaded. The drawing suggests a fractional number. You are to choose the fraction on the right which names the same fractional number as the shaded part of the drawing. Circle the letter in front of your answer choice.
>
> 
>
> 1/6   3/6   5/6   7/6   None of these
>
> 88% correct (Ginther, Ng, & Begle, 1976, p. 6)

The following example from the applications section was the fourth easiest in its subsection:

> A girl weighs 64 1/2 pounds. Her brother weighs 1/2 as much as she weighs. How many pounds does he weigh?
>
> 54% correct   (p. 7)

These results were included here to illustrate that even in the "most advantaged" schools, many students do not have a thorough understanding of fractions. The authors concluded that the poor understanding of the structure of the rational number system was due to poor instructional programs, and that until elementary and junior high school teachers could teach fractions in a more meaningful way, much of the work on fractions should be postponed to secondary school (p. 9). An alternative explanation is that the students may not have reached the level of cognitive deveopment necessary to profit from the instruction.

Efforts to help the individual student will be discussed next.

Diagnostic and Prescriptive Teaching

Diagnostic and prescriptive teaching is not new, but is emerging as an important area in mathematics education. The State of Florida has recently passed a law requiring early childhood teachers to use diagnostic and prescriptive techniques when teaching the basic communication skills. In the teaching of mathematics at all levels, the techniques seem especially appropriate.

The pioneers in using diagnosis and prescription in the teaching of mathematics were Brownell, Brueckner and Grossnickle, who did extensive work in the field beginning in the twenties and working through the forties. Interest in that effort waned during and after the war, but the preoccupation in the late sixties with disadvantaged students and the current emphasis on ensuring that minimal competencies are mastered has caused a rebirth of interest in the field.

The term "diagnosis" refers to knowing not just that the student missed the problem, but why (in the sense of "what type of error was made?"). "Prescription" means the assignment of instruction specifically designed to correct that type of error. This method of teaching has been described as shooting with a rifle, rather than with a shotgun (Glennon and Wilson, 1972, p. 283).

There have been some attempts to find the causes of "discalculia," or mathematical disability (Farnham-Diggory, 1978), including studies of brain damage (Luriya, 1968) and of the hemispheres of the brain (Davidson, Note 5). Concentrating more on psychology than biology, Scandura (1970) reviewed research in "psychomathematics." He concluded that

> there are a large number of unspecified, but crucial, "ideal" competencies which underlie mathematical behavior. These need to be identified. . . . There is also the urgent need to consider how the inherent capacities of learners and their previously acquired knowledge interact with new input to produce mathematical learning and performance. (p. 95)

These urgent needs might best be met through in-depth observations of individual students and their learnning, as is done in clinical studies.

In the meantime, many diagnosticians have taken the pragmatic viewpoint: they would like to know how students learn mathematics, but meanwhile, they try to find out specifically what students are doing wrong and to correct or remediate those errors.

Glennon and Wilson (1972) wrote a state-of-the-art paper for the 35th National Council of Teachers of Mathematics (NCTM) Yearbook, The Slow Learner. They defined diagnostic-prescriptive teaching as "a careful effort to reteach successfully what was not well taught or not well learned during the initial teaching" (p. 283). They suggested the interview technique perfected by Brownell (Brownell & Chazal, 1935) for finding out what students were doing wrong.

Lankford has used individual diagnostic interviews to survey the computational errors of seventh graders as they worked problems involving whole numbers and fractions. He tested 176 students in six intact seventh grade classes. In the interviews he directed students to "say out loud" their thinking as they computed (1974, p. 26). The percentages correct on the fraction exercises are not surprising, in view of the national assessment data; in general "the performance was much below that with whole numbers" (Lankford, 1972, p. 30). A sampling taken from that article (pp. 20-30) follows:

Table 1
Sample of Results of Lankford Study

Exercise	Percentage of Attempted Exercises Correct
3/4 + 5/2	47
3/4 - 1/2	58
2/3 x 3/5	63
9/10 ÷ 3/10	41
Which is larger, 2/3 x 5 or 1 x 5?	61

It should be noted that in the last exercise cited above, there were two choices; students could have been correct 50% of the time by chance. In fact the interviews showed, Lankford said, that sometimes students gave the correct answer for the wrong reason.

The main findings, of course, were students' thinking patterns. In addition of fractions, for example, out of 97 incorrect answers, 62 were found by adding the numerators and also adding the denominators; 10, by adding the numerators and taking the larger denominator; and 6, by adding the numerators but multiplying the denominators (p. 30). These errors might have been predicted by experienced teachers, but Lankford says, "relatively large whole numbers were a 'surprise' as when 3/4 + 5/2 = 86 . . . and 3/4 - 1/2 = 22" (p. 31). Students were adding or subtracting the numerators and denominators; the surprise lies in the manner in which the results were written. Apparently either a fraction did not have meaning as a small number for these students, or the students did not connect the meaning of a fraction with computations done on paper.

Another error demonstrates the lack of understanding of the meaning of a fraction:

3/8 + 7/8 = 11/15 (p. 31)

The answer was derived from 3 + 8 = 11 and 7 + 8 = 15; the procedure may have been a persevering pattern from the column addition of whole numbers.

And to change 3/4 to an equivalent fraction, one student reasoned, "4 times 1 equals 4 and 1 + 3 is 4, so 4/4" (p. 31). The conclusion that 3/4 = 4/4 again indicates that the student did not understand the concept of fraction, or did not connect the concept with the computation. Some students even stated that "2/3 is greater than 1" (Lankford, 1972, p. 34).

In concluding, Lankford gives pointers in the use of the diagnostic interview, suggesting that teachers can learn how well instruction has been imparted by using this method with their own students.

Glennon and Wilson (1972) also recommend Brownell's models of "ideographically oriented procedures," which they feel are effective techniques for both diagnostic and prescriptive teaching. They give credit to both Brownell and Piaget for their contributions to the development and use of idiosyncratic procedures in mathematics education, but cite as the more easily understood and readily used the work of Brownell (p. 308).

Even with Brownell's and Piaget's clinical procedures, especially frustrating and challenging are those students called "disadvantaged" or "slow learners" or "low achievers." Many teachers feel that if a formula could be found to enable their learning, all students would benefit from the formula. The only logical way this idea could be in error is for slower students to actually learn in a qualitatively different manner from the more successful students' manner. What has been discovered, if anything, about this possibility?

In Suydam's (1971) summary of research on teaching mathematics to disadvantaged students, cited earlier, she lists the following as one of the statements that can be implied from the research:

> The mathematical characteristics which distinguish disadvantaged from advantaged pupils appear to exist in degree rather than kind. That is, disadvantaged and advantaged pupils have similar abilities and skills, but differ in depth or level of attainment. (p. 13)

It is an assumption of this study that the above statement is true, and that what is learned about the learning of disadvantaged students will help the advantaged students as well.

Suydam also found that "active physical involvement with manipulative materials, which is believed to be important for all children, may be even more so for the disadvantaged" (p. 13). However, as she noted earlier, "little research has been done on this specific topic with specific sets of disadvantaged pupils" (p. 5). She concludes,

> Groups of disadvantaged pupils are not all disadvantaged in the same way. There is as much need to individualize instruction for disadvantaged students as for other groups of students. (p. 13)

Currently many compensatory and remedial instructional programs aimed at teaching basic skills do not take these individual differences into account.

There is an older study which was designed to address the problems of these students in a substantive way. Although the students were not in secondary school, but in the upper elementary grades, the spirit and method of the study and the questions asked make its review appropriate. The purpose of the study, reported by Small, Avila, Holtan and Kidd (1966), was to "explore factors related to low achievement and underachievement in mathematics education and to determine if there are individual levels of abilities in abstractive thought with respect to mathematics concepts" (p. 4).

This pilot study was an attempt to identify characteristics of low achievers and underachievers in mathematics in grades 4, 5 and 6, in hopes of finding new approaches to remediation, thereby making it possible to intervene in the processes which often lead to failures and dropouts.

Low achievers were defined as students of average IQ whose average percentile scores on all sections of a standardized achievement test were at least two deciles below their present grade placement level. Underachievers were students of average IQ whose nonmathematics scores were equal to or above their grade placement, but whose mathematics computation and concepts scores were two or more deciles below their nonmathematics percentile averages.

Small et al. (1966) used a case study approach with 12 underachievers and 11 low achievers. Each student was tested individually on two concpets, place value and linear measurement. There were three levels of questions on each subtest: concrete (the test material was a physical model which could be manipulated by the subject); semi-concrete or pictorial (materials used were photographs of real objects); and abstract (questions were asked verbally or symbolically). The report included affective results.

First, there was no consistent pattern on levels of abstraction; the ability to operate on the different levels is an individual problem and must be identified for each student.

Secondly, both the low achievers and the underachievers seemed to experience more emotional adjustment problems than did the typical student population. The underachieving student was most often a child with a large amount of anxiety and a relatively unharmonious home in which high achievement was considered importatn. The low achieving student probably needed a comprehensive compensatory program at school.

Several recommendations were made for underachievers, basically aimed at reducing their anxiety. The authors recommended a diagnosis and remediation plan involving levels of abstraction, for testing by other researchers.

The Small et al. (1966) study serves both to introduce the general field of clinical studies and to focus attention on the concrete-versus-abstract question.

Clinical Study

In the study discussed above, a "case study" approach was used with 23 subjects. Tests were administered individually. (The testing instruments are given, but the report is brief and details of the diagnostic interviews are not available.) No hypotheses were being tested; rather, the researchers were searching for factors which might be used to form hypotheses concerning low achievers and underachievers.

In many clinical studies the interview, as developed by Piaget, is used as the basic technique to gain information about children's thinking. This approach may seem unscientific to some researchers trained in standardized testing, for, as Flavell (1963) says, no two children will ever receive exactly the same experimental treatment. Even though the initial questions may be uniform,

> in the course of this rapid sequence, the experimenter uses all the insight and ability at his command to understand what the child says or does and to adapt his own behavior in terms of this understanding. (p. 28)

The same individual attention used in diagnosis needs to be used in studying the interaction of the learner with instruction, as in "teaching experiments." According to Steffe, teaching experiments share these characteristics:

They are usually long term interventions, with a small number of students. Researchers study how children learn, or the "dynamic passage from lack of knowledge to knowledge present" (Steffe, Note 6).

This microscopic attention to individual students is expected to yield much information, in contrast to traditional paper-and-pencil standardized testing, where "there is no way of knowing exactly what respondents were thinking" (Carpenter, Coburn, Reys, and Wilson, 1976, p. 137).

In mathematics education research, according to Kilpatrick, we not only want to know that certain people do better at certain things; we also want to know their characteristics, and what interaction is occurring. These things, he says, can not be learned from statistical analyses. Neither can anything be learned without sensitivity. A suggested approach is,"Let me look very intensively at a small number of people and see what is happening" (Note 7). This is the approach of a clinical study.

The next topic will be a cursory look at how others have interpreted Piaget's works and the resulting impact on mathematics education.

Related Piagetian Research

The many efforts to make sense of, and subsequently, to make use of Piaget's voluminous output can be roughly categorized as follows:

1) validation (or invalidation) studies, where attempts are made to replicate his experiments;

2) "training," or learning, studies, where experimenters test to see whether children can be taught the cognitive structures Piaget has described (classification, seriation, etc.);

3) applications or extensions of his theory and/or his methods to other situations, to education in particular. (In a sense, of course, group 2 is a subset of group 3.)

There will be no attempt here to give a comprehensive review of this work. The earlier works can be located in Flavell's (1963) definitive book on Piaget's work, and Lovell (1971a) has reviewed "twenty-five years of Piaget research in intellectual growth as it pertains to the learning of mathematics" (p. 2).

Some general comments will be made, including mention of a few relevant studies, with the primary attention given to the third group.

1) The validation studies generally support Piaget's theory, although variations are reported. Lovell (1971a) summarizes a group of these:

> By and large the stages in the development of the structures, proposed by Piaget, are found but there are differences. The age range for the elaboration of a particular structure is considerable even in children of comparable background and ability as judged by teachers or by test results. (p. 5)

Lovell states further that the situation, the actual apparatus used, and the previous experiences of the children are all variables affecting their behaviors. "It is now clear that the tasks are subtle, that the relevant ideas have to be carefully devised and that analysis has to be thoughtfully considered" (Lovell, 1971a, p.6).

2) Flavell (1963) reviews 20 training or learning studies (pp. 370-378) which pertain to the teaching of the various cognitive structures. Results were mixed; only a few reported significant differences between the trained groups and the control groups. Flavell comments,

> Probably the most certain conclusion is that it can be a surprisingly difficult undertaking to manufacture Piagetian concepts in the laboratory. Almost all the training methods reported impress one as sound and reasonable and well-suited to the educative job at hand. And yet most of them have had remarkably little success in producing cognitive change. It is not easy to convey the sense of disbelief that creeps over one in reading these experiments. (p. 377)

Just as they are difficult to induce, the conservation concepts are difficult to extinguish when actually once acquired, he says. The one study he reported in which the training group clearly outperformed the control group was one by Smedslund (1961), in which the keynote of the training procedure was the induction of cognitive conflict and the absence of external reinforcement.

Piaget's response to these efforts is usually amusement. In the first place, he does not understand why educators want to accelerate what he considers the child's natural development. Even assuming that such acceleration is a worthwhile goal, he is skeptical. Whenever he is told that someone has succeeded in teaching operational structures, there are three questions he asks. First, is the learning lasting, two weeks, a month later? "If a structure develops spontaneously, once it has reached a state of equilibrium, it is lasting; it will continue throughout the child's entire life" (Piaget, 1964, p. 184). And when the learning is achieved by external reinforcement, he asks, what are the conditions necessary for it to be lasting?

Secondly, how much generalization is possible? "You can always ask whether this is an isolated piece in the midst of the child's mental life, or if it is really a dynamic structure which can lead to generalizations" (p. 184).

The third question is, "What was the operational level of the subject before the experience and what more complex structures has this learning succeeded in achieving" (p. 184)? We must see, he says, which spontaneous operations were present at the outset and what operational level has now been achieved after the learning experience.

Recent training studies by mathematics educators have included those by Coxford (1970), Johnson (1975), Kurtz and Karplus (1979), Lesh (1975b), and Silver (1976). Some report successful training and some do not.

3) The unsettled questions just mentioned bear on the present section. In his article on psychology and mathematics education, Shulman (1970) says that Piaget's characterizations of number-related concepts have helped shape our ideas of what children of different ages might learn meaningfully. This has thus influenced some current conceptions of readiness:

> To determine whether a child is ready to learn a particular concept of principle, one analyzes the structure of that to be taught and compares it with what is already known about the cognitive structure of the child of that age. If the two structures are consonant, the new concept or principle can be taught; if they are dissonant, it cannot. One must then, if the dissonance is substantial, wait for further maturation to take place. (p. 42)

If the degree of dissonance is small, Shulman says, Piaget's theory does not recommend, but neither precludes, training procedures aimed at achieving the desired state of readiness.

Brainerd (1978) disagrees entirely with Piaget's model of learning. Since he assaults major theses, not trivial details, his arguments will be mentioned. He first takes issue with the notion that concepts will arise naturally and need not be trained. Brainerd's is a typical oversimplification of Piaget's "notion," which actually includes as requirements for this "natural" development not only chronological maturation, but also an appropriate set of experiences, providing for disequilibrium and subsequent, higher-level equilibration (Copeland, 1979; Flavell, 1963).

Further, Brainerd says that those Piagetians who do training experiments insist that the training be as natural as possible and include opportunities for self-discovery. Brainerd maintains that there is not a continuum from artificial to natural, and that there is no evidence that natural is better (Brainerd, 1978, pp. 83-84). The same original sources, in this case Piaget's theory as stated by his co-workers, can yield different interpretations. Another person, reading the same quotations Brainerd has selected (pp. 69-78), might summarize them using the phrase "relevant to the child," for example, instead of the word "natural." (This interpretation would render irrelevant Brainerd's admitted digression on Rousseau (pp. 79-84), subtitled "Is Mother Nature Always Right?") If Brainerd's recommended methods of teaching, or training, are worthwhile, whether natural or not, then of course they should be used. For example, he mentions "correction training," in which verbal feedback from the experimenter is accompanied by "a tangible reward (e.g., candy or a token) following correct responses" (p. 86). This method of teaching is not recommended by some psychologists. Not only may the reinforced behavior be extinguished when the reinforcements are removed, but also, extrinsic rewards may actually decrease the intrinsic value of the learning activity for the subject, thus doing more harm than good (Levine & Fasnacht, 1974, p. 820).

Other types of training Brainerd mentions as successful are "rule learning" and "conformity training." In rule learning, as the name implies, the students are taught a rule or rules "which may subsequently be used to generate correct responses on a concept test" (Brainerd, 1978, p. 87). In conformity training, children who missed the concept questions on pretests are grouped with children who answered the pretest questions correctly. Asked to arrive at "consensual answers," the conservers apparently convinced the nonconservers. Brainerd says that "79% of the pretest nonconservers learned all five concepts. . . . All improvements were stable across a 1-week interval" (p. 88). One must accept the statement that the 79% gave correct responses; Piaget would want to wait more than a week to see whether the children had "learned all five concepts."

In further critique, Brainerd selects three predictions he says the Piagetian theory makes. First, learning interacts with children's knowledge of to-be-trained concepts.

But, Brainerd says, few learning theories would not say that.

Secondly, preoperational children cannot learn concrete operations concepts. Brainerd says that this has been disproved (Brainerd, 1978, p. 105).

And thirdly, concepts belonging to different stages must be learned in a certain order. Brainerd says that this is a trivial outcome; the way the stages are set up, each stage includes the concepts of the previous stage (pp. 100-101). He concludes that "although we may need a readiness perspective on concept learning, Piaget's approach does not seem to be it" (p. 105).

The basic thrust of this study concerns the possibility of improving our knowledge of how students learn, or fail to learn, mathematics, the fraction concept in particular. The value of Piaget's theory in this effort, if any, will not be that it is correct and esthetically satisfying in every detail, but that it adds to our knowledge of how students learn or fail to learn, that it enriches our diagnoses of students; difficulties, and possibly, that it, eventually, inspires more successful teaching techniques. Consequently, the first and third predictions, which Brainerd finds insignificant, are not weak points in this context; the ideas might prove to be valuable to an educator attempting to sequence instruction for the student's maximum success.

In trying to refute the second prediction ascribed to Piagetian theory, Brainerd again violates Piaget's assumptions. In setting the stage for the studies that he says prove that preoperational children can learn concrete operational concepts, he states, "preschoolers should be almost completely untrainable. . . . in a sample of 3- to 4-year-olds . . . it should be safe to assume that concrete-operational mental structures are not present" (p. 96). As mentioned previously, Piaget's theory does not say what should be, but describes what has been observed. The mental structures of a child are developed individually and may not be congruent with those of his age group. It seems evident from Piaget's experiments that it is not "safe to assume" anything about a child's thinking. Brainerd cites a training study on number and length conservation with 4-year-olds, saying that

> there was clear evidence of transfer. The same subjects passed roughly 41% of the items on the mass and liquid quantity post-tests. (Brainerd, 1978, p. 100)

With the item format not available, it is not convincing that 41% correct answers represents clear evidence. In the other experiments mentioned, retention was again tested only one week after training.

If Brainerd's objection is correct, however, and Piagetian concepts can be trained, and if certain Piagetian concepts are found to be related to mathematical concepts, then the path is obvious: students should be trained in Piagetian concepts before, or in conjunction with, their mathematical instruction. Certainly many mathematics educators have seemed to heed Piaget's (1973) invitation:

> If mathematics teachers would only take the trouble to learn about the "natural" psychogenetic development of the logico-mathematical operations, they would see that there exists a much greater similarity than one would expect between the principal operations spontaneously employed by the child and the notions they attempt to instill into him abstractly. (p. 18)

Piaget optimistically says that

> one can anticipate a great future for cooperation between psychologists and mathematicians in working out a truly modern method for teaching the new mathematics. This would consist in speaking to the child in his own language before imposing on him another ready-made and over-abstract one, and, above all, in inducing him to rediscover as much as he can rather than simply making him listen and repeat.(p. 19)

Lovell has called for studies which give other than pass or fail responses, and suggests that more emphasis should be placed on careful observation of the schemes which lead to correct solutions. He says that

> such studies are likely to throw light on the nature of the schemes (in respect of mathematical ideas) available to normal as compared with dull and disadvantaged pupils. . . . The classical Piagetian structural model must be supplemented. (Lovell, 1975, p. 187)

Carpenter expressed the research need as follows:

> What is essential is the construction of good measures of children's thinking and the identification of specific relationships between performance on those measures and the learning of particular mathematical concepts. (p. 76)

Several studies have used Piaget's cognitive structures as measures of children's thinking and have attempted to relate them to mathematics learning. Those most pertinent to the study of fractions will be discussed.

Hiebert and Tonnessen (1978) wanted to extend Piaget's analysis of fractions in continuous situations to other physical interpretations. They decided to replicate the experiments with continuous models and to investigate whether Piaget's analysis applied equally well to a discrete model of fractions. Nine children, 5 to 8 years old, were given three tasks in videotaped interviews. They were asked to divide a quantity of material equally among a number of stuffed animals so that the material was used up. In the area task, a circular "pie" of clay was used; in the length task, a piece of licorice, and in the set/subset task, penny candy (four times as many candies as animals). Two children had tasks dealing with halves, three with thirds, three with fourths, and one with fifths.

Six of the nine children succeeded in discrete (set/subset) tasks; only two succeeded in both length and area tasks. The explanations offered by the authors are that discrete quantity tasks do not require well-developed anticipatory schemes, while continuous quantity tasks do. Discrete tasks were solvable by number strategies (e.g., counting); length and area tasks first required a subdivision into equal pieces.

Concerning the developmental sequence, the authors said that in area representation, some children were successful with halves and fourths, but not with thirds. In the length task, the level of difficulty corresponded with the number of parts. And in the set/subset task, no order-of-difficulty sequence was observed. The predominant one-to-one partitioning strategy was used with equal success for all fractional numbers.

Hiebert and Tonnessen (1978) conclude that the Piagetian conceptual analysis of fraction is adequate to describe the children's strategies in the length and area tasks, but not in the set/subset task. Nothing inherent in the task forces the child to use the part-whole approach, since the task can be solved by simpler strategies (counting and one-to one partitioning).

Further, they say that meaningful comparison of the discrete and continuous interpretations of fractions was not possible. They did not find generalizable identifying criteria that define a complete part-whole fraction concept across all physical interpretations. "It appears that further theoretical work involving a conceptual analysis of fraction must include psychological, as well as logical, analyses if this comparison is to be meaningful" (Hiebert and Tonnessen, 1978, p. 378).

In upper elementary school, the ratio interpretation of fractions is important. But if, as Piaget has suggested (Lovell, 1971a, p. 8) and Lovell and others have confirmed (Lovell and Butterworth, 1966), proportional reasoning is not available to children until they reach the period of formal operations, then how can they understand fractions as ratios and solve proportions? Steffe and Parr (1968) investigated the success with fractions of fourth, fifth and sixth graders who had been exposed to two curricula, one using fractions as ratios, the other, as quotients, or fractional numbers. Among the authors' conclusions were the following statements:

> Children solve many proportionalities presented to them in the form of pictorial data by visual inspection both in the case of ratio and fractional situations.
>
> Whenever the pictorial data, which display the proportionalities, are not conducive to solution by visual inspection, the proportionalities become exceedingly difficult for fourth, fifth and sixth grade children to solve, except for the high ability sixth graders. (p. 26)

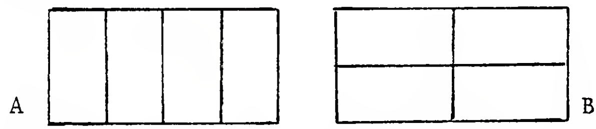
The authors raise this question, in implications for further research:

> Is it possible to construct a "readiness test" for the study of ratio and fractions in the elementary school? Such a test may have its foundation in the psychological theory of Piaget. (Steffe & Parr, 1968, p. 26)

Efforts are being made to use Piaget-type tasks in classroom diagnosis. Johnson (1980) suggests that elementary teachers can use such tasks in diagnostic interviews. Information thus gathered, along with that obtained through traditional means, "allow the teacher to develop a program based on the diagnosed strengths and weaknesses of the child" (p. 146). A set of 18 tasks are described. The two tasks that are relevant to this study are Task 17, "Meaning of a fraction," and Task 18, "Concept of a fraction."

Task 18 is an abbreviation of the cake-cuttings of Piaget, discussed in Chapter 2. However, Johnson's directions do not seem to be complete enough for a teacher's guide. The teacher may not know how to interpret it when a child cuts off two small slices for the two dolls, leaving a large portion of cake (perhaps trying to get rid of it under the table). Some sample expected answers could be provided, along with some criteria for deciding which answers exhibit what sort of understanding.

Task 17 purports to test the student's understanding of the meaning of a fraction. The materials are two 4-inch by 8-inch rectangular regions. Here are the directions:

> Take the two regions and ask the child if they are the same size. The child should be allowed to place one on top of the other to verify. Now mark region A and region B as in the diagram below.
>
> 
>
>
> Ask, "What is a fraction name for each part of region A?" "What is a fraction name for each part of region B?" Now point to a part of region A and ask if that part is the same size as one part of region B (pointing to a part of B). (Johnson, 1980, p. 164)

There may be confounding factors in the above example, relevant to the tasks used in the present study. First, an optical illusion may be operating; the horizontal length of region B may appear to be greater than that of region A, when in fact they are the same. There is also the example in Piaget's study of conservation of area, reported earlier in this paper, where the different orientation of two identical bricks changed a child's perception of the area remaining in the field. While this situation is not exactly analogous, it casts doubt as to whether the child will see the horizontal parts of region B as equivalent to the vertical parts of region A.

It seems to be assumed by Johnson that the child can conserve area. Piaget has reported protocols in which children have started with two rectangles exactly alike; having cut one into two or more parts, the experimenter asked whether there was as much room in each, the cut rectangle and the uncut rectangle. Several children maintained that there was more room in the rectangle which had not been cut (even when the experimenter put the cut pieces back together, right on top of the uncut rectangle) (Piaget, Inhelder and Szeminska, 1960, pp. 275-277). Could the markings on Johnson's rectangles function in the same way, to make the child think the area had changed? If the vertical marks changed region A, did they change A in the same way that the crossed marks changed B, if they changed B?

Care also needs to be taken in the use of vocabulary. What is the interviewer's definition of "the same size?" And does it happen to be the same definition the child is using? A tall skinny man and a short fat man might have the same mass or perhaps the same volume (or possibly even both?), but one might not say that they are the same size.

These considerations echo the comment of Lovell (1971a). He was, in turn, quoting Mayer (1961), who said that future teachers needed a "course which attempts to explore the profound aspects of the deceptively simple material they are going to teach" (Lovell, 1971a, p. 12). Certainly Task 17 was more complicated than it appeared on the surface.

Of all the models of fraction, the area model seems to be appealed to most often in schools. Therefore Taloumis's (1975) area study may bear on the teaching of the fraction concept. Taloumis wanted to standardize the reporting of abilities of primary school children in area conservation and area measurement. Also to be studied was the effect of test sequence on performance. Of the 168 children in grades 1 through 3, half did the area measurement tasks first, the other half, the area conservation tasks first. Tests were administered individually.

There were three conservation tasks. In the first one, two rectangles (index cards) were shown. As the child watched, one of the index cards was cut on the diagonal. The halves were separated, rotated and rearranged into an isosceles triangle. The child was asked whether the two shapes (rectangle and new triangle) had the same amount of space.

The second conservation task was the farm problem discussed earlier in this paper. In the third task, the congruence of two green "gardens" and the congruence of two brown "plots of ground" for flowers were established. The brown plots were placed in the gardens, and one of the plots, which was sectioned, was changed into successively longer rectangles. The child was asked whether each garden had the same amount of ground for flowers, or, if not, which one had more.

In the area measurement tasks, the child was to use as measuring devices 1-unit squares, 2-unit rectangles, and half-unit squares to compare two noncongruent shapes (the unions of rectangles). In the second task a triangle was to be compared with a polygonal shape.

Taloumis found that the sequence of presentation did affect the performance on the second group of area tasks. The conclusion includes the following:

> If area conservation tasks are administered first, the scores on area measurement tasks are increased, and vice versa. The implications for future researchers are: 1)training in area measurement may improve a child's performance in area conservation; 2) learning takes place across Piagetian tasks given in sequence. (Taloumis, 1975, p. 241)

She concludes that Piaget's theory that the ability to measure quantities is dependent on acquired concepts of conservation does not appear to be completely tenable.

Piaget's stand may not be be completely tenable. On the other hand, there might be a simple explanation for Taloumis's results: the two tasks are not all that different. In the first conservation task (C_1), for example, two plane figures are being compared. In the first measurement task (M_1), two plane figures are also being compared, but with the assistance of some smaller increments of area (unit squares, etc.).

Consider Piaget's work on area. In a conservation of area task, a child is being asked to compare the area of a rectangle with a second one which has been transformed into a pyramid. After asking the usual question about the amount of room in each shape, the interviewer says, "What if I covered it with cubes" (Piaget, Inhelder, & Szeminska, 1960, p. 281)? The child is then led to cover first one area, then the other, with the cubes, which serve exactly the same function as Taloumis's unit squares do in task M_1. For Piaget, the tasks C_1 and M_1 are both conservation tasks. Therefore it is not at all surprising that they were found to be interdependent.

When Piaget studies the measurement of area, the task is slightly different. He again asks the child to compare the areas of two polygonal regions, but using two separate techniques. With the first method, there are enough or nearly enough measuring cards to cover the area being measured. He wants to discover the age at which children will use the smaller cutouts as a middle term, or common measure. In the second method, the subject is presented a limited number of square unit cards which he must then move from one part of the surface being measured to another. The point then being observed is not simply that the child answers "equal" or "not equal," but whether the child realizes the transitivity of a common measuring term, a basic component of measurement (Piaget, Inhelder, & Szeminska, 1960, pp. 292-293).

In explaining the dependence of measurement of area on conservation of area, Piaget mentions the "harder problem," the conservation of completmentary areas, where the child must not only understand the space of "sites" which are occupied and those which are vacant, but also the reciprocal relation between the area within a perimeter and the area outside it (Piaget, Inhelder, & Szeminska, 1960, p. 291). A child may be able to comprehend the area of a thing which takes up space before the area of the "site," or space taken up. The analogous difference in one dimension was mentioned in the discussion of conservation of distance.

In addition to realizing the transitivity of a common measuring term, in order to measure area, the child must "understand composed congruence (i.e., that a number of sections taken together equal the whole which they cover)" (p. 294).

Taloumis (1975) says further that the scores showed that significant learning took place during the assessment, and that there seemed to be transfer of learning in both directions (p. 241). This result is not incompatible with Piagetian theory. For children who were transitional, the testing situation may have provided the necessary cognitive conflict, or disequilibrium, to enable equilibration at a higher level with regard to the conservation of area. In fact, the "keynotes" in Smedslund's (1961) training study, conflict with no feedback, were apparently present in Taloumis's assessment procedure.

The explorations with concrete manipulatives may have also been helpful to the children in Taloumis's study. There is considerable interest in the use of manipulatives in instruction and, more recently, in diagnosis.

Concrete Versus Abstract Modes of Presentation

The mathematics education literature has for years included recommendations that concrete, manipulable materials be used in instruction (Lovell, 1971a; NCTM, 1954; Suydam, 1970; & Swart, 1974). Shulman (1970) says that "Piaget's emphasis upon action as a prerequisite to the internalization of cognitive operations has stimulated the focus upon direct manipulation of mathematically relevant materials in the early grades" (p. 42). Of course, as Piaget uses "action," internal cognitive operations are actions. In Piaget's concept, actions performed by the subject are the raw materials of all intellectual and perceptual adaptation (Flavell, 1963, p. 82). The infant performs overt, sensorimotor actions; with development, the intelligent actions become more internalized and covert.

> As internalization proceeds, cognitive actions become more and more schematic and abstract, broader in range, more what Piaget calls reversible, and organized into systems which are structurally isomorphic to logico-algebraic systems. (Flavell, 1963, p. 82) [Flavell's emphasis]

Flavell insists that despite the enormous differences between them, the abstract operations of mature, logical thought are as truly actions as are the sensorimotor adjustments of the infant (Flavell, 1963, p. 82). Piaget's notion of development, then, is active, interactive; thinking and knowing are actions that one performs.

Flavell also interprets certain of Piaget's beliefs about education: In teaching a child some general principle, one should parallel the developmental process if possible. The child should first work with the principle in a concrete and action-oriented context. Then the principle should become more internalized, with decreasing dependence on perceptual and motor supports (moving from objects to symbols of objects, from motor action to speech, etc.)(p. 82).

It must be remembered that Piaget was not himself an educator; he provided a theoretical rationale for certain recommendations, but no practical instructions for teaching. Some mathematics educators have tried to apply strategies which would provide for active learning in the spirit of Piaget. They reason that children should be provided both concrete or manipulable objects and diagrams which could illustrate the mathematical concepts being taught symbolically. It is not clear that teachers or students always know what to do with these learning aids.

Payne (1975) reviewed research on fractions done primarily at the University of Michigan. Most of the studies that compare different instructional sequences are not germane to this study, but some do relate to the question of mode of presentation. For example, Payne says that "where meaningful approaches to operations on fractions have been compared to mechanical or rule approaches, there appears to have been some advantage for the ones that were meaningful" (p. 149). Further, he says, "when there was an advantage favoring meaningful approaches, it was usually most evident on retention tests" (p. 149). "Meaningful" and "mechanical" were not always clearly defined; however, Green's (1970) study, according to Payne, had a logical development but relied heavily on physical representations (Payne, 1975, p. 150).

Green investigated the effects of concrete materials (one inch paper squares) versus diagrams and an area model versus a "fractional part" model on fifth graders' learning the algorithm for multiplication of fractions. Since the study is not available in its entirety, excerpts of Green's summary, as quoted by Payne, will be given. Basically the approach using area was more effective, and the diagrams and manipulative materials were equally effective. Of further interest is Green's note:

> The failure in finding a fractional part of a set definitely points to the need to

> find a more effective way to teach this important concept. Particular attention should be given to overcoming the difficulty children have with the "unit" idea, relating the model and the procedure for finding a fractional part of a set, and delaying the rule until there is understanding of the concept. (Payne, 1975, p. 153)

Perhaps the difficulty alluded to is caused by the need to have logical class inclusion firmly in place for the understanding of a part-whole relationship (Kieren, 1975; Piaget, Inhelder, & Szeminska, 1960).

Payne says that Green's results were better than those of similar studies. Green's approaches all involved visual models: either concrete materials that children manipulated or diagrams of regions. Since all her retention scores were almost 90% of posttest scores, Payne concludes that

> the use of visual materials in developing algorithms has a more important effect on retention than does a purely logical mathematical development. (Payne, 1975, p. 155)

However, the use of manipulative materials did not seem to have the expected advantage in achievement. Payne says that evidently it is not a simple thing to relate a child's thought to his use of concrete materials or diagrams (p. 156).

Kurtz and Karplus (1979) undertook a training study to see whether ninth and tenth grade prealgebra students could be taught to become proficient in proportional reasoning. Manipulative materials were hypothesized to be more effective and to engender more favorable attitudes than paper and pencil activities alone. The authors' conclusions were that proportional reasoning was taught successfully, that manipulative materials and paper and pencil activities provided equal cognitive gains, but that the manipulative version was considerably more popular than the paper and pencil version (Kurtz & Karplus, 1979, p. 397).

Except for studies such as the above, the use of manipulatives in instruction has been primarily restricted to the elementary schools. An interesting result came from a study (Barnett & Eastman, 1978) of ways to train prospective elementary teachers in the use of manipulatives in the classroom. Subjects either received demonstrations only (control group) or both demonstrations and "hands on" experience with the manipulatives (experimental group). On the test on the uses of manipulative materials, the authors found no significant difference between the groups. However, the experimental group did better on the mathematics concept posttest. The authors suggest that

> a plausible explanation for this result may be that although subjects do not learn to "teach better" by actually using manipulatives, they may better learn the mathematics concepts involved. The results of several studies have suggested that many preservice elementary teachers have not reached the level of abstract operations, and hence they might need manipulative aids themselves in order to learn the mathematical concepts that they are expected to teach. (pp. 100-101)

The 1960 study mentioned earlier (Small, Avila, Holtan, & Kidd, 1960), which focused on diagnosis rather than instruction, seemed to be based on Bruner's learning theory. Mathematics educators may be drawn to Bruner's work because he often takes examples from mathematics for his own research and demonstrations. As summarized by Shulman (1970), Bruner says that the child moves through three levels of representation in learning: the enactive level, where the child manipulates materials directly; the iconic level, where the child deals with mental images of objects; and the symbolic level, where the child manipulates symbols only. This sequence, Shulman says, is based on Bruner's interpretation of Piaget's developmental theory (p. 20).

Bruner does acknowledge Piaget as "the most impressive figure in the field of cognitive development" (Bruner, 1967, pp. 6-7), but disposes of him as a psychologist:

> It is not his rather easy conception of equilibrium-disequilibrium that has contributed to our understanding of growth. Rather, it is his brilliant formal description of the nature of the knowledge that children exhibit at each stage of development. . . . But in no sense does this formal description constitute an explanation or a psychological description of the processes of growth. (p. 7)

Bruner gives his explanation of intellectual growth: By representation we translate experience into a model of the world. The nature of intellectual development is that it seems to run the course of the three systems of representation until the human being can command all three (pp. 10-12).

In his notes on a theory of instruction, Bruner says that any problem or domain of knowledge can be represented in these three ways, exemplified concretely by the use of a balance beam. A young child can act on the principles involved in a balance beam; he exhibits this by handling himself capably on a seesaw. An older child can in addition represent a balance beam for himself or others by a model on which rings can be hung or by a drawing. Finally, a balance beam can be described in words, without diagrammatic aids, or even better, by reference to Newton's Law of Moments (Bruner, 1967, p. 45). In discussing instruction, Bruner suggests that, if his analysis of intellectual development is correct (enactive through iconic to symbolic representation), then an optimal instructional sequence would progress in the same direction.

Thus Bruner's and Piaget's ideas on education are not incompatible. Their basic difference can be seen in Bruner's concluding point in his essay on patterns of growth. He suggests that

> mental growth is in very considerable measure dependent upon growth from the outside in--a mastering of techniques that are embodied in the culture and that are passed on in a contingent dialogue by agents of the culture. (p. 21)

Piaget's theory includes growth from the inside out, in interaction with the environment; Bruner holds that growth occurs primarily from the outside in.

These quite different viewpoints lead many mathematics educators to the same conclusion: that students at different levels of abstracting ability may need different types of learning materials with which to represent their models of the world. Small et al (1960) reasoned that if the learning sequence were a continuum, a child's place on it could be identified and the child could then be taught "appropriate lessons on the appropriate level of the concrete-abstract continuum" (Small, Avila, Holtan, & Kidd, 1966, p. 33). Engelhardt (1980) has reviewed some research on diagnosis and prescription in remedial clinics. He supports the use of concrete materials, but raises some questions. Should diagnosis proceed from concrete to abstract or vice versa? Are there manipulative tasks for diagnosing all mathematics concepts? What effect does student unfamiliarity with the materials have on performance (p. 35)?

Some researchers have tried to embody Bruner's theory of the learner's representations in physical models. The investigators in the Small et al. (1960) study were so carelul that in the stimuli for the iconic level (which they call semi-concrete or pictorial) they used actual photographs of the concrete objects. In other studies, some researchers have grouped diagrams with paper and pencil materials to contrast them with manipulative materials (Kurtz & Karplus, 1979). Some see concrete and visual materials as different (Green, 1970); others contrast situations where the subjects manipulate objects themselves with those where the objects are manipulated by demonstrators (Barnett & Eastman, 1978). To summarize, the use of concrete or manipulable objects by the learners themselves in an instructional setting sometimes has a positive effect on achievement or attitude or both. Exactly why, to what extent, or how it works is not clear.

Rationale

A previous section reported efforts by mathematics educators working backward from the goal, or content to be taught, to find subconcepts and subskills needed to understand fractions and to compute with them.

In considering the learner, a sampling of the cognitive development theory of Piaget was given, with examples of tasks which illustrate the child's thinking in the preoperational (stages I and II) and concrete operational (stage III) subperiods. It was reported that as children mature and have appropriate experiences, many develop concepts such as conservation of number, the ability to form logical classifications, etc., often without benefit of specific instruction in school. It was observed that for some reason there were some students, referred to as disadvantaged, who had not profited from instruction in mathematics. Their mathematical capabilities seemed to differ in degree rather than in kind.

In a discussion of the interaction of the learner and mathematics content, Piaget's work with fractions, or partitions, was reviewed. Some of the examples illustrated the fact that nothing should be assumed about a child's knowledge. Assessments reviewed indicated that, generally, some students are not satisfactorily acquiring fraction concepts. Efforts of those in the diagnostic and prescriptive field, who study what went wrong when students do not compute correctly, were discussed. The clinical study and some neo-Piagetian research was discussed, and the calls from various mathematics educators for research into the underlying structures of fractions were mentioned. Concrete versus abstract modes of presentation of mathematics content were discussed.

This section will present an attempt to take a new look at the concept of fraction, from the point of view of the student.

The Student's Notion of Fraction

If it were possible to forget the mathematical structure of the rational numbers, one could look naively at a fraction: the symbols on the page are numerals and a slash mark, written in close proximity. To meaningfully interpret this configuration of symbols, the student must have a concept of number and must know which numbers are used (nonzero numbers are used in the denominator, for example). The student must realize that the arrangement of the numerals and slash mark is important (2/3 ≠ 3/2, and 2/13 ≠ 21/3). In addition to the individual meanings of the component parts, the student must understand the various meanings of the whole symbol. (For example, 2/5 can be interpreted as 2 out of 5, 2 divided by 5, or as a ratio of 2 to 5.)

But a notion of fraction is already developing informally in the pre-school child. Words like "half," "divide," and "third" are acquired by the young child through hearing others use them, just as other vocabulary words are learned. Unfortunately, the common uses of "half," for instance, are so varied that a child may have difficulty abstracting a general meaning for the word. To a child, "half an hour" may simply mean an indeterminate time; "half a mile," a distance keyed to some fixed reference point; to "half" an apple may mean to cut it into and take out the core; and "halfwit" may mean someone stupid. These and other uses have little in common to give the developing child an understanding of the meaning of the word "half."

Nevertheless, children often do manage to acquire a correct notion of half, usually from dividing up, or partitioning, things with siblings and friends. Note that the word "half" may still not mean the same to the child as "one half," and it may not have anything to do with the numbers 1 and 2.

The child also hears what seem to be fraction words in ordinal usage, like "third base" and "fourth grade." These homonyms may be confused for years.

Some preschool children do acquire some understanding of simple unit fractions. Suydam (1970) reports that upon entering school, at least 50% of the children can recognize halves, fourths and thirds (p. 1). (In what form they are recognized is not stated.) Campbell (1975) found that 5, 6, and 7-year-olds could understand halves, thirds, and fourths, especially in concrete representations. They could put fractional parts back together to make a whole, and they could determine half of a whole. Throughout the school years, "half" has a very strong image for students (Kieren, Note 8; Hart, Note 9). Thus, a child often has an intuitive verbal concept of "half" and possibly other fractions, which is independent of formal schooling.

Concurrently, the child's conception of number has been developing. (This is quite distinct from rote counting.) During primary school, children meet the symbols used for numbers, and hopefully make a connection between the symbols and their internal conception of number. At some point, children are exposed to the idea that the numeral 5 is not always the number 5; sometimes it is 50, and sometimes 500. It depends on where it is on the paper in relation to other numerals. Some children begin not to trust arithmetic at all and try to avoid school mathematics, using their own processes when solving real life problems. This effective separation of school mathematics from real life is particularly evident when fractions are introduced.

The concept of fraction is often introduced in elementary school with an area model. But suppose a child can not yet conserve area. When the teacher says that the area is divided into 6 equal parts, the child may not comprehend it, but simply count to 6, and write down "1/6" as required. When the teacher says that 6/6 is a whole, the child may accept the fact that it happens on paper at school without making any connection to events in his world.

Consider the child for whom the operation of class inclusion is not yet in place. Faced with a figure showing 3 parts shaded out of 8, the child can see only the 3 shaded parts and the 5 unshaded parts; as soon as attention is focused on the subset (the 3 shaded parts), the greater set (the 8 parts) is "lost." Such a child often identifies the fraction as "3/5."

In the discrete model for fractions, children may be shown 3 blue chips and 4 white chips and asked, "What fraction of the chips are blue?" The child mentioned above, not able to conceive of the whole set of 7, might very well say "3/4."

In an alternate question, the child may be shown an array of 9 chips and be asked to put a string around 5/9 of the chips. The child who can not conserve number might be very hesitant to shift the chips, if necessary, to get them to fit inside the string. Or, if the child does move them, the new configuration may have no relationship in the child's mind to the original set of chips.

Suppose a number line model is used to illustrate the concept of fraction. This model is important if the child is ever expected to measure, using the jungle of marks on an inch ruler. But consider the child who thinks that the toy soldiers are closer together when a fence is inserted between them. What happens to the unit interval when little markers are introduced to show the appropriate subintervals? Is 1/2 really equivalent to 2/4, or did the extra mark somehow alter the distance from 0 to 1/2 for the child?

This model points up a difficulty (not directly related to the distance task) often observed. Some children, attempting to count up subintervals, begin at 0, counting the zero also. (This is a very interesting phenomenon; the children, when measuring, count the zero end of the ruler as 1 inch or 1 cm. Attempts to clarify the situation by using a "stepping off" procedure may not help, as the children tend to count the stationary position as 1 and the first step as 2.)

It is hoped that the child will connect the intuitive notion of half to the symbol "1/2." The child may learn to generalize this and other intuitive notions to a concept of fraction. (The fact that fractions do have multiple meanings makes this more complicated. In a typical assessment tool, for instance, 1/2 is used in two different ways in the same problem, given on p. 56 of this paper.)

The fraction concept is not complete until the child comprehends that 0 can be thought of as the fraction 0/5, for example, and that 1 is equivalent to 3/3, etc.

The above notes and ideas on a student's concept of fraction have been derived from observations made during years of teaching. Piaget's views on the child's concept of fraction (discussed on pp. 44-52 of this paper), as far as they went, seemed to be very much in accord with what had been observed. Kieren's (1975) work, reviewed earlier in this paper, was concerned with the interpretations of rational numbers as a system, including operations within the system. This study is concerned with one component of such a system, the meaning of a fraction. Three sections of the Written Fractions Test (Appendix F), sections D, E, and F, do turn out to be akin to Kieren's first interpretation (p. 17 of this paper), just symbols used in computation. The discrete model of fractions, as described in this study, is similar to his second interpretation, equivalence classes (pp. 18-20 of this paper). Although he does not specifically mention area, the area model of fraction as described here is also a part-whole model and so is also close to that interpretation. Kieren felt that class inclusion might be important in that interpretation. The number line model of fraction as used in this study is related to Kieren's (1975) sixth interpretation, rational numbers as measures (p. 22 of this paper). Again, he suggests that students need a part-whole concept. He also indicated that conservation of area and length might be related to partitive division. The Novillis (1976) analysis of the concept of fraction, discussed earlier, was also highly dependent upon a part-whole model, for which it appears that logical classification might be necessary.

These considerations led to the following conjectures concerning the fraction concept:

1) To understand an area model of fractions, it seems necessary on logical grounds that the student should be able to conserve area. With this example, some more points can be made. First, this does not necessarily imply that a student can not learn the concept of fraction without conservation of area; they might be acquired concurrently, for the study of the area model of fraction may help the student develop the concept of area (especially if the student is transitional with respect to the conservation of area). But it may mean that the teacher can not depend on appealing to the foundation of a well-defined area concept on which to base the fraction study.

Secondly, "necessary" is not to be confused with "sufficient." It is not proposed that the concept of fraction will develop automatically with the development of the Piagetian concepts; it is assumed that instruction in the specific mathematics content is still required.

2) The number line model of fraction seems to be related to conservation of distance.

3) For the discrete model of fraction, conservation of number seems to be prerequisite.

4) For all three models of fraction, logical classification, including class inclusion, seems to be necessary.

5) In order to operate with fractions, one must work with the individual component numbers, the numerators and denominators. This suggests a further need for conservation of number, already mentioned in 3.

6) To compare lengths in the number line model of fraction, the student should be able to seriate, or order, lengths.

There may be other relevant structures, but these seem to be crucial to the concept of fraction and to operations with fractions. The possible relationships discussed above have been explored in this study.

The primary sources used for the task formulations were Copeland (1979), Formanek and Gurian (1976), and Piaget, Inhelder, and Szeminska (1960). The tasks were ordered on the basis of projected difficulty.

Concrete or Manipulable versus Pictorial or Written Presentations

The mode of presentation of mathematics content may interact with learning or, in diagnosis, with understanding of content being assessed. It was important that the modes of presentation be considered in the testing of students on Piagetian concepts and on the fraction concept. The secondary school students who are in the target group usually are not provided with manipulative learning or assessment materials, except perhaps in demonstrations. Such pictures or visuals as appear usually do so in conjunction with the symbolic or written presentation. But there are differences between the various formulations of a task or test question. It was decided that the tests would each be given in two modes.

The Piaget-type tasks were presented first with manipulable objects, and then similar questions were asked, with the tested concepts presented pictorially. Those students who were operational (or in stage III, as defined earlier) should be able to perform the task correctly in either form. Students in stage I, not having acquired the concept, probably would not be able to answer either form correctly. But if there is a difference in how students respond to manipulable and pictorial representations, then the transitional students, in stage II, might be able to respond correctly in one format and not in the other. Both correct and incorrect responses are expected in transitional students. If a pattern becomes evident, it could indicate that one format is easier for some students than the other.

Similarly, two forms were planned for the fractions test, with parallel item content. One test was based on concrete materials that students could see and manipulate in answering the questions. The other test was written, very like a test on fractions that a student might take in school. The written test was thus essentially symbolic, with a few pictures or diagrams related to individual questions.

To account for the possibility that learning may go on during the assessment, half of the students were given the concrete fractions test first, and half, the written form first.

Clinical Methodology Used

The methodology used in this clinical study was different from that of experimental research. Impartiality was still observed, in that the investigator did not help or teach the subjects. However, it was necessary to pay very close attention to each individual subject for a period of time.

The purpose of the study was not simply to find out whether a student could answer a question correctly or not, but to learn about the student's thinking. Consequently, tests were administered individually. In all but the written fractions test, an interview format was used. Efforts were made to see that the student was not made uncomfortable by the testing process.

On each task or item, the initial question was uniform, but, as discussed earlier, unanticipated student responses required the investigator's insight as to how to proceed. When a student could not answer on a particular task and was in danger of becoming very frustrated or uncomfortable, for example, the line of questioning was ended or altered as gracefully as possible. Language differences, negative attitudes toward school or toward mathematics, or unfamiliarity sometimes made communication unclear. It was the investigator's responsibility to search for the student's meaning without exerting pressure on the student. Details of the methodology will be provided in the discussion of procedures.

Question 1

To explore Question 1, whether a particular level of cognitive development was prerequisite to the student's acquisition of the fraction concept, certain Piagetian concepts and certain fraction subconcepts were selected for study. Specifically, this study was designed to investigate possible relationships between:

1) Conservation of number and the discrete model of fraction;

2) Conservation of distance and the number line model of fraction;

3) Conservation of area and the area model of fraction;

4) Class inclusion and the three concepts of fraction; and

5) Conservation of number, seriation, classification, and class inclusion and overall success in the fractions tests.

The search for hypotheses included detailed study of individual student protocols.

Question 2

In exploring whether the mode of presentation makes a difference in students' performance, the study was designed to compare students' performance on:

6) Manipulable and pictorial forms of the Piaget-type tasks; and

7) Concrete and symbolic versions of the fractions test.

Again, the individual students' protocols were studied for patterns.

CHAPTER THREE
PILOT STUDY

A pilot study was conducted during the summer of 1979 at Gainesville High School, Gainesville, Florida.

Subjects

Subjects were students in the summer school Individualized Manpower Training (IMT) class, an individualized, vocationally-oriented mathematics class taught by the investigator. Six students were chosen as representative of both the summer school class and the regular academic year population for that class. Four students completed the testing program: Tim (age 15), Angie(18), Phyllis (16), and Phil (14).

Instruments

Three instruments were developed. The instrument to be administered first was a set of the six Piaget-type tasks thought to be most related to the concept of fraction. The second phase of testing included tests on fractions in two forms: one in symbolic, or written, form, and the other using concrete, or manipulable materials. Each of the three tests was intended to be graduated in difficulty within the test, with the simplest questions first. Modifications were made during the study, some after only one administration when difficulties were discovered. The findings and modifications will be detailed following the descriptions of the original instruments and procedures.

In the development of the instruments, several general principles were followed. First, it was kept in mind that Piaget had studied preoperational concepts with small children. There was evidence that some low achieving students in secondary schools are functioning at the preoperational level. Yet physiologically, sometimes legally, they were adults. Piaget's tasks had to be adapted to yield the same information without insulting the student. Instead of penny candy, the discrete objects to count were poker chips, for example. Since the testing had to be done within a school situation, there would be limitations of space, time, and materials. Thus an attempt was made to follow the spirit of Piaget's investigations, but to simplify the tasks. (The study is not intended to be a direct replication of Piaget's experiments.)

On the fractions tests, the general goal was to provide for communication of meaning between the investigator and the student. Any questions which could be answered by rote use of algorithms were avoided, at least on the concrete form. (On the written test, there could be no control over the method of solution.) To prevent non-standard administration of the tasks or careless use of the language, interview scripts were prepared with directions for the investigator, the questions to be asked written in upper case letters. Practically every word in the scripts was scrutinized for possible bias or misunderstanding, as the intent was to test concepts, not vocabulary. If a student did not understand the initial question, the investigator was to repeat or paraphrase the question, being careful not to give hints or additional information (at least until a judgement had been made about the student's understanding of the concept.)

Piaget-type Tasks

At least two different subtasks were prepared for each of the six tasks, a concrete version and a representational version intended to be analogous to the former, using pictures instead of manipulable objects. The tasks as initially administered are given in full in Appendix A, and are summarized below.

Conservation of number. In task I, in the concrete form, there were five subtasks, all using game chips. The student was asked to make two sets of chips equal in number, to decide whether sets of chips in different arrangements were equivalent, and to divide 16 chips into four equal sets. In task II, referred to at that time as the representational form, there were two subtasks, both related to whether two pictured sets were equivalent.

Seriation. In task III, the concrete form, the student was asked to put drinking straws of different lengths in order, then to insert some drinking straws into a sequence already ordered. In task IV, an ordered sequence of straws was pictured. The student was asked where a separate pictured straw would go in the sequence.

Classification. In task V, the student was shown a set of geometric objects classified by color, and was asked to group them in other ways. In task VI, the student was asked to classify various pictures cut from magazines.

Class inclusion. In task VII, once the student had successfully grouped some geometric objects, "quantifying" questions were asked to see whether the student could attend to both the part and the whole, the subset and the whole set. The task was repeated as VIII, with picures of food cut from magazines.

Conservation of distance. In task XI, the concrete version, the student was asked whether the distance between two toy men was changed when screens of different sizes were placed between them. In task X, two persons were pictured and the process was repeated, the screens being placed perpendicular to the paper.

Conservation of area. In task XI, a geoboard and rubber bands were presented. The student was shown four unit squares and asked to use a rubber band to make one shape that contained the same amount of space as the four shapes. Then shapes made of construction paper were rearranged and the student was asked about the equivalence of the room taken up by the shapes. In task XII, the farm problem, discussed earlier, was presented.

Fractions Tests

The content selected included the concept of fraction in the discrete, number line, and area models, and those operations with fractions which could be "concretized," or formulated in the concrete form as well as in the written test. This ruled out computations based on non-meaningful algorithms and those whose models would be too complex, such as the division of a fraction by a fraction. It was decided to include sections on equivalence of fractions, comparison of fractions, and addition and subtraction of fractions. The latter section involved adding and subtracting fractions with like denominators only, since equivalence was tested in a separate section, and since the algorithm for finding common denominators, etc., introduces other variables not under consideration.

The versions of the tests used in the pilot study are given in full in Appendix B (Concrete Fractions Test) and Appendix C (Written Fractions Test), and are summarized below.

Concrete fractions test. In section I, the concept of fractions, area model, small plastic rectangles were used to form larger rectangles. The student was first asked what fraction was red in three cases, then was asked to use the pieces to illustrate the meaning of fraction in three cases.

Section II was on the concept of fractions, discrete model. The student was asked to name what fraction of a set of chips was blue in three cases, then to put a string around 1/4 of a group of 12 chips, and in three cases to use chips of different colors and/or string to illustrate the meaning of fractions.

In Section III, the concept of fraction, number line model, a string mounted on construction paper was used as the concrete representation of a number line. The student was first asked to identify three fractional points on the number line, and then to estimate where on the line two fractional points would be.

Section IV was addition and subtraction of fractions, a topic difficult to represent concretely. A small box was used, with small oaktag rectangles covering the bottom. The student first had to identify what fraction was blue, then green, then what fraction of the bottom was covered by the two colors together. A take-away process was used for subtraction.

In section V, the student was asked to compare the purple parts of two rectangles made of construction paper, and then to rearrange them so that the fraction of the rectangle that was purple was the same in both.

In section VI, the student was asked to put a string around 2/3 of an array of 9 chips.

Written fractions test. This form was not formally subdivided into sections, but a group of items was parallel to each of the sections on the concrete form. Items 1 through 5 were on the area concept of fraction; in the first three, students were to identify fractions from drawings of rectangles and in two items they were asked to shade fractions of blank rectangles. Items 6 through 9 were on the discrete model, with pictures of chips. In two items they were to identify fractions shown, and in the other two, to represent fractions by circling chips. Items 10 through 13 were number line questions, the first two identifications; in each of 12 and 13, they were to place a dot at a fractional point on a number line. Items 14 through 18 were addition, 19 through 23 subtraction. Two of each were written horizontally. In items 24 through 27, the student was asked to circle either the smallest or the largest of four fractions, and in 28 through 30, to circle a fraction equivalent to a given fraction.

Procedure

All three tests were administered individually in a workroom behind the classroom. In some of the cases, the investigator still had responsibility for the class, easily seen through glass windows, and in other cases, students stayed after class for the testing. The facilities were shared with another teacher and three aides, who tried to keep distractions to a minimum; nevertheless there were some interruptions.

The administration of each of the three tests was tape recorded. In addition, the investigator made detailed written notes of students' procedures for solving problems, such as manipulating objects, making marks or diagrams, etc., and also behaviors such as frowns, smiles, and wiggles. Later a transcript was made which combined the words of the student and the details of the behavior which accompanied the words.

The physical setup consisted of a large table, with the investigator and the student sitting close enough to each other that both could reach the objects to be manipulated.

All subjects completed the Piaget-type tasks first. Then the two fractions tests were given; two students received the concrete form first and two the written first. The written test was first given on regular paper; after the second administration, items were separated and each was put on a 5" x 8" card, so that the student could attend to only one item at a time. The tasks and the concrete fractions test consisted of dialogue; the student was not required to do any writing at all. The written test was conducted essentially in silence. If students asked questions about the items, the investigator rephrased the question, sometimes suggesting that students could guess or skip the items if they did not know the answers. Tim was chosen as the first subject, because he was one of the better students in the target group. He was expected to perform successfully on all the tasks and most of the fraction items. Also, it was felt that his self-concept would not be shaken by the investigator's procedural mistakes or awkwardness in the test administration. (Similar tasks had previously been administered to younger children, but it was assumed that there might be differences in older students' reactions.)

Findings and Discussion

Observations related to the administration of the tests included the following:

It was difficult to avoid the teacher mode, wanting to instruct. Trying to be neutral sometimes resulted in overcompensation, such as dropping a line of questioning which seemed to make a student uncomfortable.

Even though care was taken to make the vocabulary as easy as possible, re-examination of the protocols indicated that the questions may not have always been clear.

Students apparently thought the question "Why?" meant that their answers were wrong.

Piaget-type Tasks

As mentioned, an effort was made to have a minimum of two versions of each task, a "concrete" and a "representational," as those designations are frequently used in the literature. There were difficulties with this which were not anticipated when the instruments were developed.

The following comments refer to the tasks as initially administered (Appendix A).

Conservation of number. In I, subtask 3, students could succeed without understanding what they were doing. They were asked to divide 16 chips into 4 equal sets. If they began by making sets of 4, they would eventually find 4 equal sets. The total number of chips used should not have equal factors.

Although it was believed that these tasks might be too easy for secondary students, two subjects did not give completely correct answers. In I, subtask 5, 12 black chips were arranged in a 3 by 4 array, and 12 blue chips in two 2 by 3 arrays. Asked "Are there more black chips or more blue chips or are they the same?" Angie replied, "More black." She did not count, but looked at them and made that judgment. This was surprising, since she correctly completed several of the subsequent tasks thought to be more difficult. She may not have been interested enough to count them, or may have thought she was supposed to guess. Phil had trouble with the pictorial (II) version. If these students had been asked to justify their answers, they might have reconsidered. However, the fact that two out of four were not completely correct justified retaining these tasks in an abridged and edited form.

Seriation. In the concrete task (III), where subjects insert straws into a sequence already ordered, Angie had trouble. She tried pairing the straws first; seeing that there were not two the same length, she finally inserted them in the sequence, still misplacing one straw. In the pictorial version (IV), she missed the correct space by one. Again, subsequent tasks thought to be more difficult, yet completed by Angie, raised questions about why she missed these, a visual problem being one possibility. In the concrete partial sequence, Phil also tried to make pairs. He did not succeed in putting the straws in order.

Classification. Two subjects were not successful with this task in either form. Both Phyllis and Phil seemed to be more interested in figural arrangements than in logical classifications based on attributes. Phyllis could make some sets (V), but did not exhaust the collection. In one attempt, there were red and yellow triangles left over; asked about them, she replied, "Them there (sic) don't go together with them." Asked if she could arrange them another way so that they would go together, Phyllis made designs (rows consisting of large circle, triangle, small circle). The same red and yellow triangles were left over, and she refused to deal with them. Phil fit all his triangles into a shape, which he referred to as a "puzzle." He would not disturb his shape after he completed it, but in answer to questions, just kept moving the circles (not included in the puzzle). The word "arrange" may have reinforced the tendencies of Phllis and Phil to make figures.

The representational version of task VI consisted of the classification of pictures cut from magazines. There were several men in various positions, a portrait of a man (frame and all), a vase of flowers, Mickey Mouse, several pieces of furniture, a sailboat, a watch, and a phone. Some of these items did not fit into obvious groups. The intent in including them was to find out how the students reasoned. For example, would they put the portrait of a man with the furniture or with the men? (Three put it with the men.)

Subjects classified both on the basis of common characteristics (all the men together, all the furniture together) and of function ("These are travel items"--suitcase and sailboat). Tim invented a whole situation to include diverse elements:

> This is a class, see. You teach them how to tell time with this [watch], and you teach them about numbers with this [telephone]. You could teach them about plants [flowers].

This was the first task that pointed up the difficulty in the intended analogies between the concrete and representational versions of the tasks. In both the conservation of number and the seriation tasks, the second version had used pictures of the same objects that were used in the concrete version.

In the classification and class inclusion tasks, subjects were asked to group geometric objects so that they were alike in some way. The objects were thin, but nevertheless prisms, with bases of triangles, circles, quadrilaterals, etc. So a "circle" was not a circle at all, of course, but a circular disk. If in fact circles or squares could have been presented, and a strict analogy were to be maintained (concrete to representations of the same object), subjects would have to be presented pictures of circles and pictures of squares. But how would these pictures be put into groups? They would have to be cut out in some fashion; therefore, there would some background, hence some shape, which would interfere with the geometric attribute being emphasized. When presented the disks and prisms, students are expected to ignore the thickness (and think of the circular disks as circles, etc.) and concentrate on the color, shape of the base, etc. Some of the students' classifications were made not by abstract geometric attributes, but on the basis of the thickness of the pieces or the material of which they were made. One student grouped by the letter stamped on the back of the plastic pieces.

It is easy to see why the "representation" classification tasks in the literature have used pictures of everyday things, such as different types of foods, flowers, or other material objects to avoid these difficulties. Consideration of this issue led to a decision to distinguish not been concrete and representational, but instead between concrete or manipulative and pictorial for the main study.

Class inclusion. The answers to this task were all invalid, because the task was formulated incorrectly. In Piaget's tasks (Piaget, 1965; Inhelder & Piaget, 1969), he used beads and flowers. The task tests whether the children understand that if $A \subset B$, then $A + A' = B$ and $n(A) < n(B)$. In one example, Piaget used 20 poppies and 3 bluebells (Piaget, 1965, p. 167). Even though certain children would agree that both poppies and bluebells were flowers, they maintained that there were more poppies than flowers. That is, they were comparing A with A', and not with B. Students in the pilot study were tested with the concrete geometric objects (VII) and then with pictures cut from magazines (VIII). Students were asked to classify the objects before they were questioned on the cardinal number of the subsets. The error in the instrument was that there were more objects in A' than in A. There was no way the student could miss the question. Tim's protocol illustrates this (investigator is denoted by I):

> I: Are there more red shapes or more circular shapes?
> T: There are more of the other red shapes.

Tim actually supplied the word "other," but it became clear that all the subjects were thinking it. In fact there were more "other" or non-circular red shapes. No information was gained.

Some interesting protocols did occur, however. Asked to group the geometric objects, Phyllis did some logical classifying, but her figural tendencies extended into the third dimension. She made stacks of objects, graduated in size, the largest on the bottom. She again had some objects left over, saying, "And these four left won't make any stack."

There were pictures of various kinds of foods for task VIII. Tim put one picture of chicken with a picture of eggs and named the set "poultry and dairy products;" another picture of chicken he put with a meat group. Chicken being in two different places did not seem to concern him as he reviewed his work.

Conservation of distance. In the concrete version (IX), two of the students realized that the toy soldiers were the same distance apart even when objects were placed between them. Phyllis was not bothered by the plane (thin, actually) obstructions, but the block of wood (7 cm wide) did alter her answer.

I: Are they close together or far apart?

P: Pretty good ways apart.

I (with low screen): Would you still say they're far apart?

P: They're half and half.

(The same reply was given with the tall fence.)

I (with block, 5 x 7 x 10 cm, the 7 cm width between the toys): Would you still say they're just as far apart?

P: They is done (sic) come a little bit closer together.

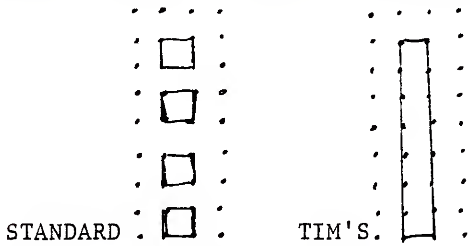
Phyllis's pattern was repeated with the pictorial version (X).

Conservation of area. Subjects' responses to this task were intriguing. One difficulty was the recurring problem not recognized when the original instrument was developed, the distinction between concrete and representational. The administration of the instrument revealed another ambiguity. Area has to do with place surfaces (two dimensional). If a plane surface is cut into pieces which can be manipulated, is the task concrete or representational? Is the task representational if diagrams of a surface are shown? Is a task concrete only if the subject manipulates the objects, or also if the investigator manipulates concrete objects, with the subject watching? If a subject manipulates pictures, is it manipulative or pictorial?

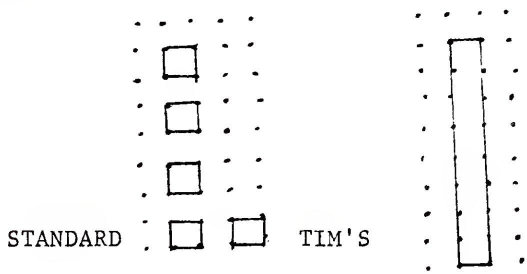
Other difficulties occurred in the questioning. Major changes were made in the area tasks after the first student was interviewed. In the initial formulation of XI, Tim was presented the geoboard first, then rectangles made of paper squares, then rectangles made of paper triangles. In the geoboard task, four unit squares in a vertical column were enclosed by four rubber bands.

> I: Can you use some rubber bands to make a shape that has just as much space inside as these four shapes?

Tim made one long rectangle, the length of the boundaries of the outside shapes, apparently ignoring the space in between the four unit spaces, as shown below.



A new question was then introduced to see if Tim could be forced to think about the spaces between the four unit squares. The same four unit spaces of area were enclosed, but an additional unit square was enclosed, not in the same column with the others. Tim started out the same way and made the same shape he had made before. Then he said, "and then one more," and added another unit of length at the bottom of his rectangle.



An easier geoboard subtask was designed. The last two subjects were given this question first:

A standard shape is shown, and then several other shapes. The student is asked, "Can you find any of these shapes that have just as much room inside as this blue one (the standard)?"

With this beginning, Angie was successful with both the four and the five units of area.

The questions with the paper rectangles were answered correctly only by Tim.

In the last area conservation subtask, XII, students were shown two construction paper "farms." On each was a model of a cow. When barns were added to each farm, one at a time, they were added in a row on the subject's farm, but scattered about on the other farm. With each pair of additions, the student was asked, "Do both of these cows seem to have the same amount of grass to eat?" Tim said "yes" for the first barn; "yes" for the second barn, and "no" for the third barn. Re-examination of the paper farms showed that there was a "stream" running through the farm; one barn had inadvertently been placed on the far side of the stream from the cow. Depending on the depth of the stream, Tim may have been correct. The farms were replaced with plain green pastures for subsequent subjects.

Angie and Phil answered correctly and unhesitatingly; the questioning was stopped at 5 barns with Angie and at 4 with Phil. Phyllis became doubtful at 2; by 4, asked if both of the cows had the same amount of grass to eat, replied, "Yours done got more (sic)."

It turned out that the area subtasks had not been presented in order of increasing difficulty. When the scores are displayed as in Table 2, a partial ordering can be seen.

Table 2
Success in Area Subtasks

	Farm	Geoboard	Paper Rectangles
Phyllis			
Phil	+		
Angie	+	+	
Tim	?	?	+

\+ Pass
? Questionable

The "questionable" scores were given due to errors on the investigator's part which made it impossible to decide whether the student had passed or not. The partial ordering seen in the table seemed to indicate that the order of presentation should be changed to farm, then geoboard, then paper rectangles.

Concrete Fractions Test

Two of the four students in the pilot study took the concrete fractions test. The following generalizations can be made:

The test was too long.

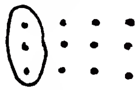
Concrete examples of addition and subtraction of fractions seemed artifical and contrived.

The sections on comparing fractions and equivalent fractions needed amplification.

Specific comments follow:

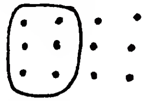
Tim was expected to achieve nearly 100% on this test. However, taking the test seemed to be a learning experience for him. On item 10, he was presented a 3 by 4 array of chips and asked, "Can you put this string around 1/4 of the chips?" He first put the string around four of the chips, but did not seem satisfied. Finally he did as shown in a below; possibly influenced by the word "circle" which had been used earlier, he hesitated and changed it to that in b.

a 

b 

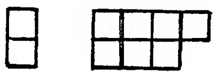
On item 13, he was asked to use chips and/or string to show what was meant by zero fourths. He said that it was "none," but was puzzled about how to represent it. The string with nothing in it was suggested by the investigator; Tim drew it. Then, on item 23, when the green parts had been removed, he was asked what fraction was green now. Tim correctly responded, "None, or zero fifths."

His incorrect answers on equivalent fraction, items 24 and 27, seemed to be due to carelessness, since he had answered similar ones correctly. However, the concept of equivalent fractions was not entirely in place. In item 29, 12 chips were put in a 3 by 4 array, and he was asked to put the string around two thirds of the chips. His solution was:



Tim may have been representing two groups of three.

Phil did very poorly on this test. He answered item 1 correctly. He also correctly answered 3/6 for item 2. But when asked, "Can you think of another way to say that?" he wrote 6/3. After that, the numerators and denominators seemed to be part-part for him, rather than part-whole. Shown 4 red parts out of 8, he wrote 4/4. In item 4, to illustrate 1/2, he placed six small rectangles together, all yellow. Asked what part was 1/2, he replied, "All the yellow ones." In item 5 he was asked to show what was meant by the fraction 2/7. He made the following arrangement of tiles:

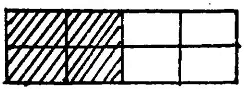


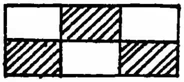
Asked which part was 2/7, he said, "The two (pointing to the 2 tiles on the left) and then the sevenths (pointing to the 7).

The only further item Phil completed correctly was item 28, where he circled 9 out of 12 to show 9/12. Since the stimulus was 9/12, there was no way for him to apply a part-part conception.

Written Fractions Test

Items 1 through 5 are concerned with the area model of fraction. Two students answered 4 out of 5 correctly; Angie missed all five. In the rectangle on the left, Tim recognized the shaded part as 1/2. But for the one on the right,

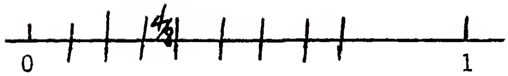




he wrote 3/6 for the shaded part. It is not known whether he realized it was also equivalent to one half.

Items 6 through 9 used the discrete model. Tim had all four correct; Angie, 3 out of 4; and Phyllis, 2 out of 4. The questions missed were related to equivalence. (Example: Asked to circle 1/4 of 8 chips, two students circled 4 chips.)

Number lines were used in items 10 through 13. Scores were: 2, 3, and 3 out of 5. All missed the first question, where they were asked to identify the name of a point on a number line (3/5). Tim did not understand how fractions represent points. In item 13, he was shown a number line with 0 and 1 labelled, and asked to put a dot about where 4/8 should be. Tim's response was:



It was as if he made 8 spaces and took 4 of them; there was a segment left over which did not get used. (This is reminiscent of Piaget's 5-year-olds who, in dividing clay cakes up among dolls, were unconcerned about the cake left over.) Also, he seemed to be labelling the interval rather than the point.

There were 10 problems in addition and subtraction of fractions (items 14 through 23). The three students scored 0, 9, and 10. It seemed that they either could do it or could not; 10 problems probably were not needed to demonstrate which was the case.

There were four items (items 24 through 27) in which students were asked to circle either the smallest or the largest. Scores were 2, 2, and 4 out of 4. The items, which were poorly chosen, were:

24. Circle the smallest: 1/9 1/2 1/4 1/6
25. Circle the largest: 1/8 1/3 1/10 1/4
26. Circle the smallest: 1/4 5/6 3/5
27. Circle the largest: 1/3 6/7 2/5

Two students answered half the items correctly using incorrect processes. Phyllis picked 1/2 for item 24 and 1/10 for 25. She was obviously associating the smallest fraction with the smallest denominator, and the largest fraction with the largest denominator.

This error pattern, combined with disregard for the numerator, yielded the correct answers for items 26 and 27.

Angie answered 24 and 25 correctly. However, her choices for 26 and 27 (5/6 and 1/3) showed that she was using the rule, "smallest denominator = largest fraction, and largest denominator = smallest fraction, never mind the numerator."

There were 3 questions (items 28 through 30) on equivalent fractions. Scores were 0, 1, and 3. No particular pattern was discernible in the errors. There were other equivalence questions in the test, items 2, 3, 7, and 9.

General Observations

In general, the trial of the instruments in the pilot study revealed the following:

The average length of time needed to administer the tasks was about 40 minutes (the range being 30 to 60 minutes). Both fractions tests could be given in about 40 minutes (range, 20 to 50 minutes).

Students expressed anxiety about the time they were taking, and about whether their answers were correct.

When the questions were very easy, the students seemed to expect a "trick."

Several of the questions needed to be rephrased or clarified.

There were some difficulties with the materials used.

Resulting Modifications

Some general changes will described, and then some specific modifications in the instruments will be detailed.

First, it seemed necessary to ask more probing questions to uncover the bases of students' responses. Since the question "Why?" seemed to induce anxiety in some students, the following alternative formulations were substituted:

"How do you know that?"

"How can you show that?" and

"How did you decide?"

There were several other student concerns that needed to be addressed. Standard prefatory remarks were prepared, to be read before each instrument was administered, to ensure that the students were made aware of the following notions:

There is no time limit.

Some of the questions have more than one correct answer.

Some of the questions are very easy; there are no trick questions.

Some of the questions are not so easy; it is always okay to guess or to skip a question or to say I don't know.

There will be no feedback until the end. Questions like "Why?" or "How do you know?" do not mean that the answer is wrong; they are asked to get an explanation of the student's thinking.

Vocabulary, materials, and procedures were streamlined as discussed below.

Tasks Instrument

To make it less cumbersome and more coherent, the instrument was changed to six tasks numbered with Roman numerals. Each included subtasks, at least one concrete or manipulative, identified by A, and at least one pictorial, B. This revised instrument is given in full in Appendix D. Both the wording and the tasks were modified in some places.

Conservation of number. In subtask 3, students were asked to divide 20 chips (instead of 16) into four equal sets.

Classification. The word "arrange" was avoided; students were asked to "group" or "put together" objects that were alike in some way.

The students were involved earlier in the task, to make sure that they understood the question. Instead of being told that the geometric shapes were grouped according to color, the students were asked to identify the attribute by which they were being grouped by the investigator. If necessary, there to be led to that first observation. It was intended that if students did not answer correctly, it would not be because they did not understand what they were being asked to do.

Class inclusion. The number of objects was modified so that the subset under consideration was greater in number than its complement. In A, the concrete version, there were 5 red circles out of 8 red geometric objects. In B, pictorial, there were 8 pictures of fruit out of 12 pictures of food. The quantifying questions were standardized, so that the determining questions were:

In A: "Are there more red shapes or more circles?" and in B: "Is there more food or more fruit?"

Conservation of area. In general, "room" was added as a synonym of "space." The tasks were rearranged in what seemed (in the pilot study) to be the order of ascending difficulty. The farm problem was first; questioning was to go up to 4 or 5 barns, depending on whether the student seemed to be frustrated or bored.

The geoboard tasks were next. An additional problem was inserted (Appendix D, VI A, 2b, p.249). The student was presented as standard a 2 x 2 square, and asked to use rubber bands to make a shape that had just as much room inside as the standard. It was thought that this exercise might be easier than the one with four individual units.

The paper rectangles were presented last.

Fractions Tests

The order of presentation of topics was made more clearly parallel to the order of presentation of the Piaget-type tasks thought to be related. The concept of fraction models were therefore presented discrete first, then number line, then area.

The two fractions instruments were made more obviously parallel to each other; analogous sections were named the same letters of the alphabet. (No attempt was made to have individual items congruent, however.)

Both tests were shortened.

Questions concerning equivalence of fractions were separated from the other sections, at least as far as scoring was concerned. That is, not understanding equivalent fractions would not affect the student's score on concept of fraction, area model, for instance. However, when there was an alternate answer (1/2 or 3/6, for example), a question was added, such as, "Can you say another way what fraction of the chips are blue?"

Concrete fractions test. The revised instrument is given in full in Appendix E.

Each new section was begun with an illustrative example to ensure that the student understood the question. There was a risk that some learning might occur even during the example, thus affecting the diagnosis. It was felt that this would not be the case for students who already had acquired the concept, nor for those students who had no notion of the concept at all. But for those students who were transitional with respect to the particular concept, the example might make a difference. It was decided that this risk would be taken, rather than the risk that a student who had acquired the concept might miss the question through misunderstanding the directions.

In the administration of the test, some physical modifications proved to be necessary. The paper rectangles and squares for items 25 through 27 were found to be competely unmanageable; they would not stay where they were put. These items were discarded.

The box arrangement used in items 19 through 24 was unsatisfactory. The items were deleted. In fact, any meaningfun concrete model of addition and subtraction of fractions seemed too cumbersome; in the revised form no concrete embodiments of these operations were included.

<u>Written fractions test</u>. The instrument is given in full in Appendix F.

Since students were to take this test essentially independently (as they might take school tests), the prefatory statement included instructions to the students to ask questions about any words or questions they could not read or understand.

As mentioned, the revised instrument was organized in sections to parallel those of the concrete fractions test, with one exception: the section on the addition and subtraction of fractions was retained in the written version. The revised order for both fractions tests was as follows:

A. Concept of fraction (discrete model)

B. Concept of fraction (number line model)

C. Concept of fraction (area model)

D. Equivalent fractions

E. Comparing fractions

F. Addition and subtraction of fractions (written test only)

The difficulties with comparing fractions, mentioned previously, were corrected by better choices of wrong answers, in the hope that correct answers would thus convey understanding of the relative size of fractions. The error patterns detected in the pilot study would no longer yield correct answers.

CHAPTER FOUR
MAIN STUDY

Subjects

Subjects were 25 students enrolled in compensatory mathematics classes for tenth, eleventh, and twelfth grades at Eastside High School, Gainesville, Florida.

Students had been placed in the compensatory classes using the following criteria:

Tenth graders had scored 30th percentile or below as ninth graders on the Stanford Test of Academic Skills (TASK) (1973) in the spring of 1979.

Eleventh graders had scored 28th percentile or below on the TASK in the spring of 1979, or had either failed the Florida functional literacy test or had failed to pass 3 or more of the standards of the Florida basic skills test in October, 1979.

Twelfth graders had either failed the Florida functional literacy test or had failed to master 3 or more of the basic skills standards in October, 1979.

Three teachers of these compensatory classes were asked, "Do you have any students who have trouble with fractions?" Each of the teachers immediately named one or more students in each class period. These students were used in the study, with a few exceptions. Occasionally a student was absent, ill, or otherwise unavailable; volunteers from the class were substituted. This study was conducted in May, 1980; several of the students' names were listed on wall charts in their classrooms as having passed their basic skills during the academic year.

The participating students' ages ranged from 14 to 18 years, with a median of 16 years. There were 10 girls and 15 boys.

Instruments

The instruments, revised according to the insights received during the pilot study and described in Chapter 3, are given in full in the appendices.

Tasks

The tasks (Appendix D) were numbered I through VI, each task containing at least one concrete or manipulative formulation in Section, and at least one pictorial version in Section B. The task titles are:

I. Conservation of number

II. Seriation

III. Classification

IV. Class inclusion

V. Conservation of distance

VI. Conservation of area

Fractions Tests

The revised fractions tests had five analogous sections as listed in Chapter 3, designated by the letters A through E; in the written test there was an additional section on the addition and subtraction of fractions (Section F). The concrete fractions test is given in Appendix E and the written fractions test in Appendix F.

Procedure

In each of the classes from which students were taken, the investigator introduced herself and made a statement about the study. The substance and flavor of the statement are contained in the following summary:

> I want to know how students learn mathematics. Some of you may be subjects in an experiment if you like. I would like to get inside your head and watch you work, but I can't so I will ask you questions. Your answer is not as important as how you get it. When I ask "Why?" it does not mean that your answer is wrong; I just want to know how you arrived at your answer.

The investigator then assisted for a class period as an aide, to learn students' names and to become comfortable with them. Care was taken not to let students know that they had been selected for the study by the teacher. Since most were eager to participate in the study, the investigator attempted to give the appearance of choosing randomly from among the volunteers.

Testing

The series of three tests was administered to each subject individually in a conference room with a round table. Testing conditions were nearly optimal. There were few interruptions during testing sessions. The interviews were tape-recorded, and notes were made of subjects' behaviors and expressions. If a student did not understand what was asked, the item was reread. If it still did not seem clear, the item was paraphrased. All students' questions about how to respond ("circle," "show," etc.) were answered.

At least two hours were allowed per student for completion of the testing. Two students took a little more than two hours. Each student was given the tasks first; on his or her next available school day, the two forms of the fractions tests were given. The concrete form was given first to 12 students, and 13 received the written test first.

The written fractions test was designed to be similar to work students normally do in school; therefore no oral questions were asked of the student. Visible behaviors were recorded, and students' work on scratch paper was saved. The items were presented one at a time, on 5" by 8" cards. Occasionally a student wanted to return to a previous question; this was allowed and noted.

Scoring

Similar procedures were used for scoring individual items or subtasks and for scoring the overall sections, representing tasks or major subconcepts.

Items or subtasks. On the individual items of the two interview tests, the student who received a Pass (+ on the score sheet) gave the correct answer firmly and could justify the answer.

The student who did not pass (blank on the score sheet) either missed the question entirely, or gave the right answer but a wrong reason.

A third possibility was that the student was "transitional;" a student who is in the process of acquiring a concept sometimes makes a statement, but is not sure about it. The student may change the answer, or can be swayed by questions such as "How do you know that?" Such responses were scored as Questionable (? on the score sheet). The student also received a score of Questionable if the investigator was not sure (i.e., if there was reason to believe that a wrong answer could be due to a mistake in the test administration, or to a careless error or a visual problem, rather than a conceptual error).

Overall sections. To be considered operational, or receive a Pass on one of the six Piaget-type tasks, the student had to be successful on both forms (A, concrete or manipulable,; and B, pictorial).

To receive a Pass on an overall section of the fractions tests, the student had to have all the individual items in the section correct. (In a few cases, exceptions were made, when an item score of Questionable was not consistent with the student's pattern of responses.)

A student received a score of Questionable on a section if most of the items in that section were correct.

Planned Data Examination

In an exploratory study, statistical tests are often not performed. However, the magnitude of information makes it necessary to summarize the data in some way to see whether patterns are evident. Certain findings are anticipated, whether specific hypotheses are stated or not; unexpected findings may also be useful in providing new insights and questions for further research.

Three kinds of data examination were planned for this study: summaries of data, by frequencies and/or percentages; the use of the Walbesser contingency table (Walbesser & Eisenberg, 1972) to display such possible relationships as might exist; and the qualitative study of individual protocols.

The data summaries were to include tabulations in various forms of both correct and incorrect responses.

The Walbesser contingency table was to be used to investigate the possible relationships proposed in Chapter 2. Those having to do with Question 1 of the study were:

1) Conservation of number and the discrete model of fraction;

2) Conservation of distance and the number line model of fraction;

3) Conservation of area and the area model of fraction;

4) Class inclusion and the three concepts of fraction;

5) Conservation of number, seriation, classification, and class inclusion and overall success in the fractions tests.

The possible relationships having to do with Question 2 were:

6) Manipulable and pictorial forms of the Piaget-type tasks; and

7) Concrete and symbolic versions of the fractions tests.

In addition to the above examinations of data, the individual student responses were to be studied qualitatively, for thinking processes and underlying understandings and misunderstandings from which might arise questions for further research.

CHAPTER FIVE
FINDINGS OF MAIN STUDY AND DISCUSSION

Findings

The first kind of data examination to be presented includes summaries of overall data. In some cases reference is made to more detailed data available in appendices.

Secondly, the seven possible relationships proposed in Chapter 2 will be displayed in Walbesser contingency tables.

Third, such generalizations as can be inferred from the study of individual student protocols will be stated.

Overall Data

The students were generally not very successful. Specific findings follow:

Tasks. The percentage of students successful on each task is given in Table 3. The greatest percentage of

Table 3
Percentage of Students
Successful on Tasks

	Task	Percentage of Students
I.	Conservation of number	36%
II.	Seriation	44
III.	Classification	8
IV.	Class inclusion	8
V.	Conservation of distance	56
VI.	Conservation of area	0

success was on conservation of distance; about half the students were successful on this task. No student passed the conservation of area task.

Table 4 gives the tabulation of the students' performances on the individual items (the tasks instrument is given in Appendix D). Students are identified by the first letter or the first two letters of their names. Items marked "S" in the table were skipped because the student had failed a prerequisite question; items marked "S^e" were skipped accidentally by the investigator. This form of data display does not yield quantitative measures; however, the tabulation of students' scores may be looked at as a scatter diagram in which patterns and trends may be illuminated more readily than in the numerical data that summarize the scores. In an inspection of Table 4, one can see the lack of success in the last section of the table, the area task. Among individual items, item 4 of subtask IV B can be selected as very difficult for the students. This item was the determining question for the pictorial class inclusion subtask. The percentage of successes for the items in Table 4 are given by subtask in Appendix G.

The percentages of students succeeding on the two forms of the tasks, A, concrete or manipulable, and B, pictorial, are given in Table 5. No general pattern appears in the table; in four tasks students were more successful

Table 4
Success on Subtasks of the Tasks

Subtask	C	S	J	O	Z	M	E	G	P	B	M	B	A	L	W	M	J	J	D	D	A	D	V	C	R
	a		o			a				a	i	e	u			g	a	i	o	e	m	a		h	
I A 1	?	+	+	+	+	+	+	+	?	+	?	+	+	+	+	+	+	+	+	?	+	+	+	+	+
2	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3a		?	+	+		+	+	?		+		?	?	?	+	+	+	?	+	+	+	+	+	?	?
b		+	+	+	S	+	+	+	S	S^e	S	S	+	+	+	S^e	+	?	S^e	+	+	+	+	+	+
4		+	+	+	?	+	+		+	+	+		+		+	+	?	?	?	+	+	+	+		+
B 1	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+	+	+	+	?	+	?
2	?	+	+	+	+	?	+	+	+	+		+	?		+	+	+	+	+	+	+		+	+	+
II A 1	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
2		+	+	+	+		+	?		?	?	+	+		+	+		+	+	+	+			+	
B	+		+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+		+	+	+	+
IIIA 1	+	+	+	+	+	+	+	?	+	?	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
2		+		+		?	?	+	?		?		+		+	?	+	?		+		+	+	+	?
3	?			?	?		?		+	?		S			+		+	?					+		+
B 1	+	+	+	+	+	+	+	+	?	?	?	?	?	?	+	?	+	+	?	+	+	+			+
2	?	+	+	+	+	+	+	+	?	?	?	?	?	?	+	?	+	+	?	+	?	+			?
IV A 1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	?	+	+	+	+	+	+
2a	?	+	+		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
b	+	+	+		+	+	+	+	+	+	+	+	+	+	+	+	+		+	+	?	+	+		+
3	+		+			+	+		+		?	+			+	+	+	+			+		+	+	+
B 1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	?	+	+	+	+	+	+	+
2	+	+	+	+	+	?	+	+	?	+	+	+	+	+	+	+	+	?	+	+	+	+	+	+	+
3a	+	+	+	+	+	+	+	+	+		+	+	+	+	+		+		+	+	+		+		+
b	+	+	+	+	+	+	+	+	+	+	?		+	+	+	?	+		+	+	?		+		+
4							?					+					+								
V A 1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
2	+	+	+		+	+	+		+	+		+	+	+	+	+	+	+	+	+	+	+	+	+	+
3	+		+		+	+	+		+	+	?	+	+	+	+	+	+	+	?	+	+	+	+	+	+
4	+		+		+	+	+		?	+	?	+	+	+	+		+	+		+	+	+	+	+	+
B 1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
2	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+		+	+	+	+		+
3	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+		+	+	+	+		+
4	+		+	+	+	+	+		?		+	+	+	?	+		+	+		+	+		+		+
VI A 1	+		+		+		+			?		+			+		+		+	+	+		+	+	+
2a																?									
b							?										?								
c						+	?								?										
d															?					+					
B 1					+		+				?						+						+		
2							?		?		?	+					+								

+ Pass; ? Questionable; S Skipped; S^e Skipped (error)

Table 5
Success on Concrete and Pictorial Forms of Tasks

Tasks		Percentage of Students Passing	Questionable
I. Conservation of number	A	48%	32%
	B	64	24
II. Seriation	A	58	8
	B	88	
III. Classification	A	20	44
	B	40	48
IV. Class inclusion	A	56	4
	B	8	8
V. Conservation of distance	A	72	4
	B	64	4
VI. Conservation of area	A	0	4
	B	4	8

A Concrete or manipulable; B Pictorial

on the pictorial version, and in two tasks, there was a greater percentage of successes on the concrete version.

Fractions tests. The overall performance on the fractions tests was very poor. As mentioned previously, a score of Passing on a section was obtained only by answering all the items in that section correctly, and very few students did that. Only one section (A) was passed in both forms, and by only three students (12%). Table 6 lists the percentage of students passing each of the sections of the two fractions tests.

Table 6
Percentage of Students Successful on Sections of Fractions Tests

Section		Concrete Pass	Concrete Questionable	Written Pass	Written Questionable
Models of Fractions	A Discrete	16%	32%	12%	20%
	B Number line	0	0	0	4
	C Area	12	44	0	16
D Equivalent fractions		0	0	4	0
E Comparing fractions		0	0	0	4
F (Written only) Adding fractions				32	8
Subtracting fractions				28	8

The individual students' scores on the two fractions tests are displayed also, with students listed in the same order as in Table 4. In Table 7, the scores on the concrete fractions tests are tabulated. (The instrument is given in Appendix E. Items A, 2b and 3b, and C, 2b and 3b, were on equivalences and thus were not necessary for success on sections A and C. Therefore in inspecting this table for

Table 7
Success on Items of the Concrete Fractions Test

Item	Ca	S	Jo	O	Z	M	E	G	P	Ba	Mi	Be	Au	L	W	Mg	Ja	Ji	Do	De	Am	Da	V	Ch	R
A. 1		+	+	?		?	+		+	+		+					+		+					?	
2a		+	+			?	+	S	+	+		+					+		+				?	+	?
b					S			S		+		+				?	+					S			
3a			+			?	?	S	+	+		+	+				+		+				?	+	?
b								S		+		+					+					S			
4		+	+	?		+	+		+	+	+	+		+	?		+		+	+		+	+	+	+
5		+	+			+	+	S	+	?		+		+			+		+	+		+		+	+
6		+	+					S	+			+					+			+					+
7			+			+		S		?	+	+					+		+	+					+
B. 1	+			+	+		+		+			+			S*		+	+				+	+		
2							+				S				+								?		
3		+					+										?	+							
4				+							?						+								+
5								S		?											?				
C. 1	+		+		+	+			+	+		+	+	+	+	+	+		+	+	+	+	+	+	+
2a	+		+		+	+			+	+		+	?		+		+		+		+	+	?	+	?
b		S	+			+		S				+					+			+					
3a	+		+		+	+			+	+		+	+	+	?		+		+	+	+	?		+	?
b			+			+		S		+		+	+				+								
4		+	+	+		+	+			+			+	+	+		+		+	+	+	+	+	+	+
5	?	+	+		+	+	+	S	+	+		+	+	+		+	+		+	+	+	+		+	+
6		+	+					S	?			+							?			S*		+	
D. 1		+	+	+	+	+	+	S	+	+		+		+	+	+	+	+	+	+		+	+	+	+
2			+					S	?		?				?			+							
3								S							?		+								
4				?	+			S							?										
5			?		?		?	S	?								+			+					
6		+	?				+	S	?			+				?			?						?
E. 1								S																	
2	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S

\+ Pass
? Questionable
S Skipped, because student failed a prerequisite question
S* Skipped through investigator error

patterns, these four lines should be ignored.) The relative lack of success in B, the number line concept of fraction, as compared to the other two models is apparent in the table. There were no successes in E, Comparing fractions, and very few in D, Equivalent fractions. The percentage of successes on the items of the concrete test are given by section in Appendix H.

Table 8 gives the tabulation of successes on the written fractions test (instrument given in Appendix F). Inspection shows again the relative lack of success in B, Concept of fraction, number line model, as compared with that in the other two models of the fraction concept, A and C. Students were least successful on E, Comparing fractions. The comparative success of students at using algorithms to add and subtract fractions (section F) may be noted. The percentage of successes on this test are given by section in Appendix I.

The data in Table 9 are organized in two groups by the sequence in which students took the two fractions tests. The concrete test was given first to 12 students. The percentage of those 12 students who were successful on the various sections of the two tests are listed on the left side of Table 9. There were 13 students who took the written test first. The percentages of those 13 who were successful are on the right hand side of Table 9.

Table 8
Success on Items of the Written Fractions Test

Item	Ca	S	Jo	O	Z	M	E	G	P	Ba	Mi	Be	Au	L	W	Mg	Ja	Ji	Do	De	Am	Da	V	Ch	R
A. 1			+				+					+	+	+	+	+	+		+	+				+	
2			+									+					+								
3	+	+	+	+	+	+	+			+	+	+	+	+	+	+	+		+	+	+	+		+	+
4			+	+	+	+	+			+	+	+	+	+	?	+	+			+		+	+	+	+
B. 1															+										
2	+	+	+			+	+	+			+		+	+					+	+	+	+	+		
3		+	+				+		+		+			+					+						
4					+	+					+														
C. 1			+			+	+			+		+	+	+	+	+	+			+				+	
2		+	+		+	+	+			+		+	+	+	+	+	+		+		+		+	+	+
3			+		+	+	+	+	+	+		+	+	+		+	+		+					+	+
4					+		+	+	+		?		+	+					+			+		+	
5				+							?				?										
D. 1			+				+		+	+						+						+			
2		+	+	+					+	+			+	+		+	+			+	+				
3		+		+	+	+	+		+	+		+	+	+	+	+	+		+	+	+				
4			+	+						+			+			+	+		+	+	+	+		+	
E. 1										+		+					+			+					
2					+					+					+					+					+
3		+			+				+								+								
4	+								+		+						+						+		+
5																									
F. 1			+		+		+	+	+	+			+			+	+			+	+				
2		+	+			+	+	+	+	+				+		+	+			+	+				
3			+				+	+	+	+			+	+		+	+		+	+					+
4			+	+	+		+		+	+		+	+	+			+			+			+		
5			+	+	+		+		+	+		+	+	+		+	+			+			+		
6			+	+			+		+	+						+	+			+					

\+ Pass
? Questionable

Table 9
Percentage of Students Successful
on Sections of Fractions Tests by Test Sequence,
Concrete-Written and Written-Concrete

Section		Order Given							
		Concrete[a]		Written[a]		Written[b]		Concrete[b]	
		Pass	Questionable	Pass	Questionable	Pass	Questionable	Pass	Questionable
Models of Fractions	A Discrete	25%	17%	25%	25%	0%	15%	0%	53%
	B Number line	0	0	0	0	0	8	0	0
	C Area	8	50	0	17	0	15	15	38
D Equivalent Fractions		0	0	17	0	0	0	0	0
E Comparing Fractions		0	0	0	8	0	0	0	0
F (Written only) Adding Fractions				50	17	15	0		
Subtracting Fractions				33	17	23	0		

[a]Percentages based on 12 students [b]Percentages based on 13 students

If the percentages of Passes and Questionables were pooled for each of the two sequences in Table 9, the totals for the concrete test would not seem very different (42, 0, 58, 0, and 0%; and 53, 0, 53, 0, and 0%). The data on the written test are somewhat different. The percentages of Passes on the written test are greater for those who took the concrete test first. The pooled totals of Passes and Questionable scores are also generally higher on the written test for those who took the concrete test first:

Concrete first: 50, 0, 17, 17, 8, 67, and 50%;

Written first: 15, 8, 15, 0, 0, 15, and 23%.

<u>Possible Relationships. Questions 1 and 2</u>

In a Walbesser contingency table (Walbesser & Eisenberg, 1972), as shown in Figure 1, the prediction of a hypothesized higher order behavior as being dependent on a hypothesized lower order behavior is tested by noting the frequency patterns of Pass and Fail.

		Lower Order or Prerequisite Behavior: Fail	Lower Order or Prerequisite Behavior: Pass
Higher Order Behavior	Pass	W	X
	Fail	Y	Z

Figure 1. Walbesser Contingency Table

Cell W indicates successful acquisition of higher level behavior and failure to acquire the lower order or hypothesized prerequisite behavior.

Cell X of Figure 1 indicates the successful acquisition of the higher level behavior and the relevant lower order behavior.

Cell Y indicates failure to acquire the higher level behavior and failure to acquire the relevant lower order behavior.

Cell Z indicates inability to exhibit a higher level behavor after acquisition of the relevant subordinate behavior.

Of the four possibilities, only W contradicts the hypothesized dependency. Theoretically the number of students in cell W should be zero if the lower order behavior is actually prerequisite to the higher order behavior. Entries in cell Y are not contrary to the hypothesized dependency, but give no indication as to which behavior the student will acquire first.

In the planned examination of the relationships proposed in Chapter 2, it was found that many of the contingency tables had large numbers of students in cell Y; they were not successful at either level of behavior. (For these displays, to be considered successful on a particular task or subconcept, the student had to receive a passing score on <u>both</u> forms, concrete or manipulable and pictorial or written.) The relationships are considered in the order given in Chapters 2 and 3.

1) Conservation of number and the discrete model of fraction

The contingency table is shown in Figure 2. The numbers of students on and to the right of the diagonal tend to support the proposed relationship, and the 1 in cell W tends to contradict the relationship.

		I--Conservation of Number	
		Not Passed	Passed
A Discrete Model of Fraction	Passed	1	2
	Not Passed	15	7

Figure 2. Task I and Fractions Section A

2) Conservation of distance and the number line model of fraction

No student was successful in the number line model of fraction. As shown in Figure 3, the hypothesized relationship is not contradicted.

3) Conservation of area and the area model of fraction

All 25 students were in cell Y, not successful at either the area task or the sections on the area concept of fraction. No information about the relationship was gained.

		V--Conservation of Distance	
		Not Passed	Passed
B Number Line Model of Fraction	Passed	0	0
	Not Passed	14	11

Figure 3. Task V and Fractions Section B

4) Class inclusion and the three concepts of fraction

No student was successful with all three concepts of fraction, or even with two of them. The table is shown in Figure 4; no information is gained.

		IV--Class Inclusion	
		Not Passed	Passed
A, B, C Three Models of Fraction	Passed	0	0
	Not Passed	23	2

Figure 4. Task IV and Fractions Sections A, B, and C

Since three students passed the section on the concept of fraction, discrete model (A), an additional table was made for Section A and the class inclusion task. It is shown in Figure 5.

IV--Class Inclusion

A Discrete Model of Fraction		Not Passed	Passed
	Passed	1	2
	Not Passed	22	0

Figure 5. Task IV and Fractions Section A

5) Conservation of number, seriation, and classification and overall success in the fractions tests

No student was successful on all three tasks or all sections of the fractions tests; therefore all 25 students would have been in cell Y.

6) Manipulable and pictorial forms of the Piaget-type tasks

Table 5 did not seem to reflect a trend in success on concrete versus pictorial forms of the tasks. To investigate possible relationships, all six tasks were displayed in contingency tables, using the concrete version as the hypothesized lower order or prerequisite behavior. Tasks I, II, and III had 7, 11, and 8 in cell W (Figures 6, 7, and 8, respectively), thus disproving dependency of the pictorial on the concrete for these tasks as formulated.

In tasks IV and V (Figures 9 and 10, respectively), there may be a trend for the concrete version to be easier.

		A--Concrete: Not Passed	A--Concrete: Passed
B Pictorial	Passed	7	9
	Not Passed	6	3

Figure 6. Task I, Conservation of Number, Concrete and Pictorial

		A--Concrete: Not Passed	A--Concrete: Passed
B Pictorial	Passed	11	11
	Not Passed	0	3

Figure 7. Task II, Seriation, Concrete and Pictorial

		A--Concrete: Not Passed	A--Concrete: Passed
B Pictorial	Passed	8	2
	Not Passed	12	3

Fibure 8. Task III, Classification, Concrete and Pictorial

B Pictorial		A--Concrete: Not Passed	A--Concrete: Passed
	Passed	0	2
	Not Passed	11	12

Figure 9. Task IV, Class Inclusion, Concrete and Pictorial

B Pictorial		A--Concrete: Not Passed	A--Concrete: Passed
	Passed	2	14
	Not Passed	5	4

Figure 10. Task V, Conservation of Distance, Concrete and Pictorial

B Pictorial		A--Concrete: Not Passed	A--Concrete: Passed
	Passed	1	0
	Not Passed	24	0

Figure 11. Task VI, Conservation of Area, Concrete and Pictorial

Figure 11 yields no information about the relationship of concrete and pictorial forms of Task VI, since 24 students were in cell Y.

7) Concrete and symbolic versions of the fractions tests

Very few sections of the fractions tests were passed. Figure 12 shows a contingency table for section A, the concept of fraction(discrete model) for concrete and written versions, with the concrete used as the hypothesized lower order form. This proposed dependency is not contradicted.

Similar contingency tables would show 25 in cell Y for both B, concept of fraction (number line model), and E, comparing fractions; these are not displayed.

		Concrete: Not Passed	Concrete: Passed
Written	Passed	0	3
	Not Passed	22	0

Figure 12. Section A, Concept of Fraction (Discrete Model), Concrete and Written Forms

Figure 13 shows the table for Section C, the area model of fraction. The proposed relationships of concrete and written versions is not contradicted.

		Concrete: Not Passed	Concrete: Passed
Written	Passed	0	0
	Not Passed	22	3

Figure 13. Section C, Concept of Fraction (Area Model), Concrete and Written

Students did not perform very well on D, Equivalent fractions, etiher. Two students were able to find written equivalent fractions on the written test but not on the concrete test, as Figure 14 shows.

		Concrete: Not Passed	Concrete: Passed
Written	Passed	2	0
	Not Passed	23	0

Figure 14. Section D, Equivalent Fractions, Concrete and Written

Qualitative Data from Student Protocols

Among the general findings revealed by detailed study of the student protocols were the following, which are explained and documented further in the discussion:

1. Many of the students, even graduating seniors, exhibited severe deficiencies both in Piagetian concepts and in the concept of fraction.

2. Some students could perform rote algorithms without understanding.

3. In spite of precautions, vocabulary was a problem; some students have non-standard meanings for some words and phrases.

4. Some students have apparently not had the experiences necessary for development of the mathematical or Piagetian concepts studied.

5. Students were cooperative, interested, and motivated to learn and to succeed.

6. Learning may have been occurring during the testing sessions.

Discussion

The most obvious finding of this study is that 25 secondary school students were generally not successful in demonstrating understanding of concepts usually thought to be acquired in elementary school. Other findings indicate that it would be a mistake to think that they cannot acquire these concepts. The findings will be discussed in the order in which they were presented in the first section of this chapter.

Overall Data

This was an exploratory study; the summaries of data served as starting points for the intensive study of the individual protocols. Consequently most discussion of findings is deferred to the next two sections. There are some general comments which can be made, and some questions which can be raised for the upcoming discussion.

Tasks. During the administration of the tasks instrument, behaviors of students in the various stages of development, as described by Piaget and his colleagues, were manifested in the behaviors of students in this study. (Furthermore, similar behaviors occurred in the concrete fractions test, which was also given in interview form.) That is, students for whom a concept was firmly in place were quite confident and ready to defend their answers. Students who had not acquired a concept guessed, skipped the question, or sometimes made an incorrect response with just as much confidence as those who were correct. The faces of the transitional students expressed doubt, confusion, and disequilibrium, sometimes resulting in what seemed extreme discomfort for them.

The overall poor performance on the tasks may be explained by the theory. As discussed in Chapter 2, an important element of Piaget's theory is that in addition to biological maturation, a child needs interaction with the environment, in experiences appropriate to the child's level of cognitive development. A child is presented with a problem of interest but not solvable at the present stage of development; disequilibrium occurs; through the process of equilibration, cognitive equilibrium is restored at the higher level necessary for accommodation of the problem and its solution. Piaget emphasized that children could accommodate only those content areas to which they were exposed. Apparently the students in the study were "socialized" in sports, music, art, etc., but not in mathematical concepts or some structures of knowledge. They were physiologically mature; however, many of them had apparently not had experiences necessary for their cognitive development. This possibility will reoccur in discussion of the student protocols.

Referring to Table 3, which summarizes the percentages of students successful on each of the six tasks, it may be noted that the order of difficulty was not quite as expected. Conservation of distance seemed to be easiest for these students. No particular reason for this was discovered. Conservation of area, involving two dimensions, did prove to be the most difficult task as predicted. The fact that tasks III and IV, involving logical classification, were so difficult called for further examination.

Table 4, which shows successes on the various subtasks, is a tabulation of individual student scores. Examination of this table produced questions about individual test items (why only 2 students answered item IV B-4 correctly, for example) and about patterns of individual students.

Table 5 displays success on the tasks by concrete or pictorial forms. In the first three tasks, students were more successful on the pictorial version. The fact that the pictorial version was always given second raised the possibility that learning was occurring during the presentation of the concrete form of the task. For tasks IV and V, the trend was reversed, markedly on task IV. (In task VI, comparison is probably not meaningful due to poor performance.) This exposed the need to reexamine the two forms of each task for other factors besides the form of representation.

Fractions tests. Table 6 gives the percentage of students successful on both forms of the fractions tests. Some gross comparisons can be made. From this table and from the score sheets (Table 7 for the concrete test and Table 8 for the written test) it is clear that students were not very successful on B, Concept of fraction (Number line model); D, Equivalent fractions; and E, Comparing fractions.

The lack of success on the number line model of fraction as compared to that on the other two models of fraction was explored further. While no student passed either section on the number line model, some students answered individual items correctly. Table 10 shows the average percentage of success per item in the sections representing the three models of fraction. The lack of competence with the number line model of fraction is again demonstrated.

Table 10
Average Percentage of Students
Correct per Item by Model of Fraction

Model	Concrete Version	Written Version
A. Discrete	39%	52%
B. Number line	14	25
C. Area	56	43

In considering this finding, it was noted that even a cursory look at elementary school textbooks reveals a preponderance of area and discrete models of fraction, and few number line models of fraction. Greeno (1976), cited earlier, had been consulting current textbooks when he developed his fraction models. He produced three models: area, discrete, and algorithmic, but no linear model. Students may simply not have had the exposure necessary for learning about the number line model of fraction.

It is not maintained that instruction has not been provided on equivalent fractions and comparison of fractions. Study of the protocols may explain the poor performance on these topics.

In Table 8, the number of Passes in the items of section F, Adding and subtracting fractions, was noticeable in comparison to the large number of blank spaces in other sections. In particular, a closer look was indicated for those students who had more success in section F than in the sections testing the concept of fraction.

As stated previously, 12 students took the concrete fractions test first, and 13, the written test first. Table 9 showed the percentage of successes on the fractions tests by sequence of presentation. There seemed to be an advantage on the written test for those who took the concrete test first. There did not seem to be a trend in the other direction; that is, there did not seem to be an advantage on the concrete test for those who took the written test first. This indicates that there may have been learning occurring during the administration of the concrete test. If that was the case, the factors in effecting this learning may have included the following:

1) the test was conducted individually;
2) questions were asked orally;
3) students responded orally;

4) many questions were open ended;

5) no feedback was given;

6) concrete materials were used; and

7) students were actively engaged in manipulating the materials.

The theoretical base for this study, presented in Chapter 2, would not preclude any of these factors having an effect on learning; specifically, factors 3, 4, 5, 6, and 7 would be expected to help produce learning.

Protocols were examined for evidences of learning.

Possible Relationships, Questions 1 and 2

This exploratory study was not designed to give definitive answers but to expose patterns or trends and to form questions for future research. Consequently no attempt was made to compute reliability ratios or other hierarchy validations. Contingency tables were used to display such trends as might exist. This form of data examination is more suitable for study of heterogeneous groups. In this study there are often large frequencies in cell Y (subjects who succeed at neither the hypothesized higher order or the hypothesized lower order behavior); in these cases no information is gained about the proposed order relationship. This form of data display does assist in producing further questions, however.

Question 1. Consider the finding, for example, concerning the first proposed relationship, conservation of number and the concept of fraction, discrete model. The frequencies displayed in Figure 2 were:

In cell Y, passing neither the conservation of number task nor the section on the discrete model of fraction, there were 15 students.

In cell X, passing both, there were 2 students. Frequencies in this cell are consistent with the proposed order relationship.

In cell Z there were 7 students; these passed the conservation of number task, but not the section on the concept of fraction. Occurrences in this cell do not contradict the hypothesized order relationship; the students have acquired conservation of number but have not yet acquired the concept of fraction. If the ordering relationship is valid, these 7 students are presently at a "teachable moment" with respect to the concept of fraction.

There is one student in cell W. Ben did not pass the conservation of number task, yet passed the discrete fraction section. According to the analysis presented in Chapter 2, this seemed logically impossible. Reexamination of the protocols suggested a possibility not previously entertained: If the conservation of number _is_ prerequisite to the discrete concept of fraction, and if the student is transitional with respect to the task, then perhaps learning could take place during the task assessment, and furthermore, perhaps the newly acquired concept could transfer to the fraction test administered on the following day. Both tasks involved manipulation of chips.

Ben's performance on the task (I) was as follows: He answered the first two questions correctly. On the third question (A-3), 20 chips are placed on the table.

I: Can you divide these chips up into 4 equal sets?

B: (Starts grabbing chips; makes 5 sets of 4) Like that?

I: Do you have 4 equal sets?

B: Not equal, cause there's 5 here.

I: Can you make it so that you have 4 equal sets?

B: (Takes one set of 4 off the table)Like that right there.

I: What if you had to use those too?

B: (Reaches for the bag of chips) Get some more chips.

I: No, this time you can't get any more. Just divide those that you have into 4 equal parts.

B: Four equal parts?

I: Yes.

B: I really don't know what you're talking about.

I: Well, suppose there were 4 people and they were going to divide those chips up. . . pretend it's money.

B: Oh, and I was going to divide it up between 4 people?

I: Yes.

B: (Puts out four single chips, and then deals the rest out, one at a time) Like that right there?

I: Are those 4 equal sets now?

B: Uh huh.

Although he finally succeeded, this response was scored as Questionable. He did not pass the next subtask:

In A-4, 12 black chips are arranged in a 3 by 4 array 12 blue chips in two 2 by 3 arrays.

I: Are there more black chips or more blue chips or are they the same?

B: There's more blue chips.

I: Okay, why?

B: There's just more.

In B, the pictorial version of the task, two rows of chips, 6 red and 6 blue, with the blue chips spread into a longer row, were pictured. Ben said that they were the same number.

I: How do you know?

B: Because I counted them.

And in B-2, 18 blue chips are pictured in a 3 by 6 array and 18 chips, in two 3 by 3 arrays.

I: Are there more blue chips or more red chips or are they the same?

B: (Hesitates) They're the same.

I: How do you know?

B: Cause I did (counting sets of 3) one, two, three, four, five, six. . . I knew there was 3 in a row.

The next day on the concrete fractions test, Ben readily answered the first three questions, which called for recognition of 3/8, 2/6, and 3/6 of a group of chips. On A-4, an array of 10 chips is presented.

> I: Can you put this string around 3/10 of the chips?
>
> B: (Hesitates, mumbles. Puts string around 3; then changes, puts it around 7, then changes back to 3)

The margin contains scores of both "ok" and "?"; Ben answered the remaining questions in the section correctly, so the Passing score was recorded.

Ben may have been learning during the tests. Or his unsuccessful scores may have been due to a communication problem; perhaps he had the concept but did not understand the directions.

As previously mentioned, no information was gained about possible relationships 2, 3, 4, and 5. Since 3 students were successful on the discrete model of fraction, a contingency table was made to check for the relationship of this model with class inclusion, hypothesized in Chapter 2 to be necessary for all models of fraction. That table was given in Figure 5. Cell Y contained 22 students; cell X, 2; and cell Z, none. None of these frequencies contradicted the proposed relationship. But again, there was one student in cell W. John passed the section on the discrete model of fraction but not the class inclusion task.

John's overall performance was one of the best in the group. The concepts he knew, he knew very well. One of the investigator's comments, written during the conservation of distance task, said:

> John is a particularly good example of how they answer when they know it; they act as if you are a complete idiot, for asking such a dumb question.

On the concrete fractions task, he correctly answered all the questions on the discrete and area models of fraction, none on the number line model; the investigator's note on section B says, "John apparently has not been exposed to this before."

The classification tasks seemed unfamiliar to him also. On III, a collection of plastic circles and triangles are first grouped by color by the investigator. John correctly named the groups by color. Asked to group them another way "so that they go together," John did not notice size or shape, but made one group which contained:

> A large yellow circle
> A small red circle
> A small green triangle
> A small blue triangle

The remaining objects he grouped by color. Asked to name the groups, he named the above collection "Mixed." In the next subtask, students are asked for still another grouping. John's new effort was essentially the same as the first. He seemed worried about his performance, and asked what was the purpose of it and what the investigator was going to do with the tape cassette.

In III B, students are to classify pictures of ordinary objects. John proceeded very slowly (as compared to his work on other tasks), but succeeded in grouping the pictures (furniture and telephone, he named "the household group," etc.).

Task IV begins with classifying 8 red geometric shapes, including 5 circles, 2 squares and 1 triangle. The notes on this protocol say:

> (Slow; seems to want to put one large circle with one small circle in a pattern. Finally puts all the circles together, the squares together, the triangle alone.)
>
> I: What would you call these groups?
>
> J: Round group (circles), square group, triangle group.

John correctly answered the quantifying questions on this subtask. On the last one, he seemed confident:

> I: Are there more red shapes or more circles?
>
> J: (Laughs) More red shapes.
>
> I: Why?
>
> J: Cause everything on the table is red.

In B, the pictures were of food, 8 of fruit and 4 others. Asked to make some groups of foods that were alike, John put 6 of the fruits together; the strawberry and pineapple he called "the dessert group." The investigator's procedure in these cases was to say:

Okay. Suppose we combined these two
groups (putting the fruit groups together).
What would you call it then?

This was done in order to have the fruit together for the next questions.

John was one of 15 students who followed this pattern:

I: If I took away all the food would there be any fruit left?

J: No.

I: If I took all the fruit away would there be any food left?

J: Yes.

I: Is there more food or more fruit?

J: More fruit.

No explanation for this contradiction presents itself. John did not receive a Passing score on this task.

It seemed obvious that the exercise of classifying was unfamiliar to John. The contradictory responses on A, the concrete version, and B, pictorial version, are puzzling, and are considered further later.

John's performance tends not to support the proposed relationship between class inclusion and the discrete model of the fraction concept.

Question 2. Proposed relationship 6 concerns the concrete or manipulable and pictorial forms of the tasks. The intent to make the instrument graduated in difficulty, coupled with the belief that the concrete version would be easier, renders this relationship difficult to examine: all the students received the concrete form first. It now appears likely that learning may have been occurring during the presentation of the concrete form, thus making students more successful on the second of the subtasks. In addition, there may have been other confounding factors related to the instrument itself, but not recognized until the instrument had been administered. These factors sometimes tended to add to the strength of the proposed relationship and sometimes to detract from it.

I--Conservation of number. The concrete and pictorial formulations of this task were not analogous.

In section A, chips are being manipulated. It is established that two sets of chips are equal in number. Then one set is spread out, or otherwise rearranged. The student is then asked, "Now are there more here or more here or are they the same?" There are basically three possible responses:

1) "They are the same; you have neither added any nor taken any away." (The student who is not sure will usually regress to response 2.)

2) The student counts both sets, then says that they are the same.

3) The student is influenced by perception and identifies the longer row as being more numerous. (The student who is transitional will have doubts and progress to the second response.)

Trying to render an analog of this task into a pictorial format yielded tasks like the following:

> Present a picture of two equivalent sets of chips, but with one set arranged differently, and ask: Are there more here or more here or are they the same?"

Since no manipulation of the spots on the page has occurred, the student is limited to responses 2 and 3. Simple counting is all that is necessary; logic is not required. After this analysis, it was no longer expected that the order of difficulty would go from the concrete to the pictorial, as these subtasks were formulated. The contingency table given in Figure 6 shows a frequency of 7 in cell W, confirming that form A is not prerequisite to form B.

<u>II--Seriation</u>. Figure 7 shows that the concrete form of the seriation task used was not prerequisite to the pictorial form used. A search for other factors produced one possible reason for students' relative success on the pictorial subtask:

In the concrete form, the student was asked to seriate, or order, 10 straws; and to place within a series, 5 additional straws.

In the pictorial version, the student was asked where 1 straw should go within a series.

Perhaps in the concrete version there were simply more chances for error (a Passing score was not given if even one straw was out of order).

III--Classification. In this task there may have been learning occurring. This task illustrates a decision made to ensure that mathematical concepts were being tested instead of language. As mentioned previously, to make sure that students understood what was being asked, they were given an example before being asked to classify themselves. The objects were classified by color, and students were led, if necessary, to identify the attribute by which the objects were grouped, and also to label the groups. This procedure may have helped some students to learn the concept being assessed.

When a concept is firmly in place, together with the appropriate vocabulary, students do not usually need an example; when they do not have the concept, the example does not seem to help them. In the case of the transitional student who is on the verge of acquiring the concept, the instruction may have changed that student's score from Questionable to Pass. This risk was chosen rather than the risk that a student might not answer the question correctly because of failure to understand the directions.

Here is a protocol that shows how a student was helped.

Germaine has been asked to say how the objects were grouped.

G: You took all the triangles and the circle, you put them in each group.

I: What groups would you call these, if you had to name them? What would you call these (Pointing)?

G: The circles.

I: The whole group.

G: Uh.

I: What about this one? (No response) What about this one? (No response) Well, what color is it?

G: Red.

I: All right, so couldn't you call this the red group?

G: Yes.

I: What would you call this one?

G: Blue. (etc., naming the rest correctly)

On the second subtask she was asked to group them another way.

G: Don't have to be the same?

I: So that they are alike in <u>some</u> way.

G: (Arranges in columns, one long column of circles and one of triangles; successfully names both groups.)

It was found that of 11 students who had Questionable scores on A, 5 had Passing scores on B. The frequency pattern of students' scores prompted a modification of a contingency table to display this information conveniently. Such a table for task III is shown in Figure 15. For this illustration, in contrast to the contingency tables used previously, no ordering of difficulty is postulated. The task on the left is simply the first administered, and the subtask on the top is the second administered. It is suggested that movement upward and to the right might

Subtask B

		Not Passed	Questionable	Passed
Subtask A	Passed	1	2	2
	Questionable	2	4	5
	Not Passed		6	3

Figure 15. Frequency Distribution of Scores on Classification, Subtask A and Subtask B

indicate that learning from Subtask A was transferred to subtask B. For example, those students receiving a score of Questionable on A may have been transitional; those 11 students are shown in the middle row of the table. It is seen that 2 of them did not pass B, 4 were still questionable on B, but 5 of them moved into the Passed column for B. The bottom row shows that of 9 who did not pass A, 6 moved to Questionable on B and 3 Passed B. This general movement toward better scores on B could indicate that B is simply easier (as in the case of task I, conservation of number). If that possibility can be discarded, it might indicate that learning is occurring across the two subtasks.

There is a possibility that the content, rather than the mode of presentation, may have been a factor in making B easier. That is, those students not familiar with geometric shapes and not used to noticing geometric attributes may have been more comfortable classifying the everyday objects in B.

Magic had had no difficulty with tasks I and II, but did have trouble with classification. The investigator's note says:

> In classifying it seems his language is what is weak (ability to describe things and use the language). I don't think he has had experience in classifying before. Because he learned during the test. i.e., he did better on the class inclusion test.

To document this judgment, the appropriate protocols are given.

Magic succeeded in naming the groups by color in A-1. Asked in A-2 to "group them another way so that they go together," he formed the following groups:

a. small circles
b. large circles
c. large triangles
d. small triangles

His difficulty came in naming the groups.

I: Tell me what you have there.

M: Red, blue, yellow and green mixed colors.

I: What would you call that whole group (a)?

M: 1, (counting) . . no, wait, . . . it's small; (Pointing to d) whatcha call this, I guess medium small; (b) large, (a) medium. . . Wait! I messed up. . . (discovers letters on backs of pieces and groups by letters). Time out!. . . Put L, L, L (grouping); M group. . . E group. . . Y group. . . Them's mixed, I mean, yellow, and this here'll be yellow and this here'll be red. (There are still red ones in other places) (Naming): Circle..uh, M group.

I: Why didn't you just call it a circle group?

> M: (Laughs; names, pointing to each circle) Circle group, circle group, circle group? Right? (Pause) They is the same shape. . Yeah. Like this is the same shape as that, cept it's bigger. . . (referring to the triangles). . . they're the same shape.

Subsequently the initial classification on task IV (A-1) was done with dispatch by Magic:

> I: Now you need to tell me what you've got there.
>
> M: This different shapes.
>
> I: So what are these (pointing)?
>
> M: Circles, (Pointing to the others) and the triangles, and the squares.

As startling as the fact that he could not do it in the beginning is the fact that he could learn it in such a short time. His work in classifying the pictures was not completely satisfactory, but he did not seem to have the same degree of confusion in thinking of names. A group consisting of a boat, a man in a boat, and Mickey Mouse he named "having fun. . . on a beach or something like that."

IV--Class inclusion. Figure 9 shows the contingency table for class inclusion. The frequencies tend to support the proposed relationship. The results are puzzling, however. The pictures to be classified in B were foods, very common ones. Therefore, unfamiliarity with the subject matter was not a problem. With each student, it was established first that all of the pictures together were food, and the identified subset was fruit. The determining question was, "Is there more food or more fruit?"

There were 14 students who could answer the analogous question correctly when it concerned circles and red shapes, and 11 of them could also answer these two questions:

> If I took away all the fruit would there still be food left?
>
> If I took away all the food would there still be fruit left?

Yet all but 2 responded that there was more fruit than food. Apparently the pictorial form was more difficult for these students.

Reevaluation of the task materials produced one alternate hypothesis; it concerns the number of objects. There were 5 circles out of 8 red shapes. There were 8 pictures of fruit out of 12 pictures of food. Perhaps the size of the collection made it harder to hold the larger set in mind.

V--Conservation of distance. Students were most successful on the conservation of distance task. As shown in Figure 10, there were 18 students correct on the concrete subtask; 4 of those missed the pictorial version. There were also 2 students who passed the pictorial version and not the concrete. The differences in the frequencies are not very large. Reexamination of the tasks reveals that the tasks seem to be analogous. No conjecture has been formed about why this task was easiest for these students.

VI--Conservation of area. Since 24 students could not conserve area in either form (Figure 11), no information is gained about the relationship between the concrete and pictorial versions. One student passed the pictorial version and not the concrete version. Jackie was not fooled by B-1, where squares making up a rectangle were rearranged into another shape. She responded that the shape still took up the same amount of room.

I: How do you know?

J: Because they're the same number of blocks. Just put in a different shape.

But on the geoboard exercise in A beforehand, she had fallen into the common trap of counting the pegs instead of counting the unit squares of area.

For Jackie and for other students, unfamiliarity with the geoboard may have been a problem. Thos who have a firm grasp of a concept might be able to abstract it and apply it to a variety of situations; to those for whom the concept is shaky, the physical context may be important.

Only two students had had prior experience with the geoboard; Martha was one of them. She could not conserve area; on the farm problem, which was easily completed by 13 of the students, Martha went astray at the second barn. Yet she solved one of the geoboard area problems (2c), and was close to being correct on 2a, 2b, and 2d. Only one other student was correct on any of the four geoboard exercises.

Experience or lack of experience with the materials may be more relevant to success on a concept test than expected.

Fractions tests. Figures 12 and 13 show examples of the proposed hierarchy, with the concrete form of the fractions test as lower order and the written test as higher order behavior. The hierarchy is not contradicted for the sections on the discrete and area models of fraction. However, as Figure 14 shows, there were 2 students who passed the written section on equivalent fractions while no student was successful with the concrete section. In spite of efforts to make the written test as meaningful as possible, it seemed that there were questions which could be answered by the use of rote algorithms. This topic is amplified later.

In summary, this component of the study was not very helpful in lending support to hierarchical relationships. It was very successful in generating questions; many of them are considered in what follows.

Qualitative Data from Student Protocols

The six primary findings of the study of protocols were listed earlier in this chapter. First was a discouraging general finding:

1. Many of the students, even graduating seniors, exhibited severe deficiencies both in Piagetian concepts and in the concept of fraction.

Next were listed three findings, each instrinsically interesting, but all special cases of 1:

2. There were students who could perform rote algorithms without understanding.

3. In spite of precautions, vocabulary was a problem; a fourth of the students had non-standard meanings for some words and phrases.

4. Some students have apparently not had the experiences necessary for development of the mathematical or Piagetian concepts studied.

The last two findings listed were encouraging, providing a note of optimism upon which to end this report:

5. Students were cooperative, interested, and motivated to learn and to succeed.

6. Learning may have been occurring during the testing sessions.

Before these findings are elaborated, some general comments about the study will be made. The subsequent discussion of the specific findings listed above will not follow the order in which they are presented. Supporting data for findings 2, 3, and 4, and part of that for finding 6, are subsets of the data which support finding 1. To avoid repetition, these specific data will be given first. Discussion of finding 1 will follow, so that other, not as easily categorized, information can be included there.

General comments. Piaget worked with young children whose minds were not yet cluttered with all the information and misinformation that comes at them from different sources, including school. Before the study was undertaken, it was not known whether the interview method would reveal the true thinking of the older students or would yield what they thought they were expected to say. There are instances which illustrate the confusions that occur when students attempt to apply poorly understood fraction concepts. The Piaget-type tasks were different from work that these students normally do in school; most of the task questions seemed to cause them to give attention and to ponder. They knew that no answer was stored; therefore the tasks represented problems to be solved. For the fractions questions, students seemed to feel that they were supposed to know the answers if they could recall them. It seems reasonable that the students' thinking was probably revealed more accurately on the tasks than on the fractions tests.

Worries about students' resenting the use of manipulable materials as childish were unfounded. Students' behaviors supported the investigator's belief that when the adult acts as if the task at hand is worthwhile, students will generally regard it in the same manner.

The interview process exposed the complexity of the deceptively simple questions. There is much data gained by this in-depth process of focusing on a very small portion of content and on one student at a time. All the potential information and questions from these 25 sets of interviews can not be reported and interpreted in this paper. In some cases, the effort to keep the interviews standard was inhibiting. Nevertheless, incidences of fascinating phenomena that were described in Piaget's work were witnessed.

Finding 2. Rote algorithms. As Brainerd mentioned in advocating rule learning, students can be taught a rule or rules "which may subsequently be used to generate correct responses on a concept test" (Brainerd, 1978, p. 87). In this study some students could also use rules to generate some correct responses. One rule many of them seem to have learned was the relationship

$$a/a = 1.$$

On the written test, 64% selected 5/5 as the fraction that had the same value as 1. Yet on the concrete test, when asked to use chips to show what was meant by the fraction 5/5, only 32% were successful. Only 16% used the rectangles to illustrate the meaning of 3/3. Here is an example of the misunderstanding on the concrete fractions test (item C-6):

> (Martha had just successfully used 2 red and 5 yellow rectangles to demonstrate the meaning of 2/7.)
>
> I: Can you show what is meant by 3/3?
>
> M: (Hesitates; puts out 3 yellow and 3 red)
>
> 
>
> Three thirds is one whole. Can't you do one whole?

I: Okay, if somebody came in and asked you where is the 3/3, what would you say?

M: Here's the red, 3 red ones (putting fingers on 3 red), and 3 yellow ones (putting other hand on yellow) equals to a whole one.

The configuration made to show 3/3 was a very common one, produced by 15 of the students (60%).

This is the protocol of another student who had succeeded in demonstrating 2/7:

I: Can you show what is meant by 3/3?

B: (Keeps whispering 3/3 to himself. Gets 3 red, 3 yellow; then puts some back. Arranges 1 red, 2 yellow; hesitates; keeps whispering to himself. Changes to 6 yellow; then changes to 3 red, 3 yellow again.)
I think I got it. . . Cause 3/3 is 3 over 3 and that's just like a half. One over one. . I think. Yeah, this is it.

Paradoxically, Barry had made the same configuration about three minutes before in correctly answering C-4, showing what was meant by the fraction 1/2. And just before that, shown the same configuration by the investigator on C-2, Barry had correctly identified the red portion as 3/6 of the figure. But Barry could reduce fractions mentally, and was the only student who correctly answered all of the written questions on D, Equivalent fractions.

In the discrete model of fraction, Zelda was asked to use the chips to show what was meant by 5/5 (A-6). In a typical error, she produced two sets of 5 chips, and wrote on her scratch paper the following:

$$\frac{5}{5} = \frac{1}{1}$$

Leon made a similar error on A-6. But his confusion was evident before that item. Here is the protocol from A-3:

I: Now what fraction of the chips are blue?

L: Uh, equal out to . . . one whole?

I: Can you say that as a fraction?

L: Right. It could be like three thirds, . . . out of three thirds. . . which equals one whole.

It becomes clear that students are responding to the equality of the numerator and the denominator, without regard to the meaning of each of the numbers in the fraction. These students seem to have learned a rule which could sometimes be used to generate correct answers, but it did not have the correct meaning for them.

On the written test, some students could use algorithms to compute with fractions, even though they could not demonstrate understanding of the concept of fraction. There were six students who computed all six of the addition and subtraction problems correctly. Of those six students, not one could:

put a string around 3/4 of a group of 12 chips;

put a pencil point "about where 1/4 would be" on a string number line;

put a dot to show "where 4/8 would be" on a number line on paper;

identify the point 3/5 on a number line;

shade 2/5 of a rectangle; or

put in order, from smallest to largest, these four fractions: 3/4, 1/16, 1/2 and 3/8.

Students may retain, if imperfectly, whatever they were studying the week before, or whatever the teacher emphasized most, even applying where it does not apply. On both fractions tests Germaine passed only one section: adding fractions (which her class had been studying the week before the test). The subtraction problems caused her some difficulty, though. Her solutions were:

$$3/7 - 1/7 = 4/7 \qquad \begin{array}{r} 7/8 \\ -\ 5/8 \\ \hline 5\ 1/8 \end{array} \qquad \begin{array}{r} 1/3 \\ -\ 1/3 \\ \hline 33\ 1/3 \end{array}$$

Finding 3. Vocabulary differences. Significant vocabulary differences were found in the study of the protocols. Mike's illustrates a possible confusion pointed out in Chapter 2, the fraction word and its homonym, an ordinal word. Mike has been shown a rectangle made from 3 red tiles and 2 yellow tiles.

I: What fraction of the large rectangle is red?

M: The third.

I: Excuse me?

M: A third. . . two thirds. . . you want me to tell you the fraction?

I: The part that's red.

M: The third part.

Next, with 3 red and 3 yellow tiles, he also said, "the third;" for 4 out of 8, he said, "the fourth."

In some cases, a student saying "No" was actually answering the question affirmatively. This confusing protocol shows the different meaning that some words have for Debbie. On the conservation of distance task, she has just been asked if the toys are still "just as far apart."

D: No.

I: Why?

D: They haven't moved; there's just a fence between them.

I: (Confused) So would you say they're just as far apart as they were, or not?

D: No, I wouldn't; no, they're the same length (shaking head no).

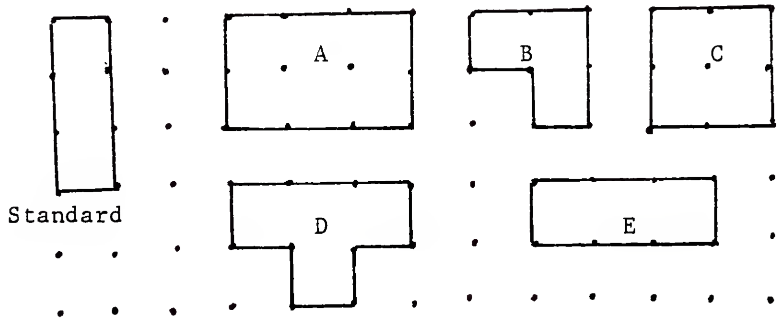
I: Let's see, maybe I'm not asking the question right. (Repeat the whole thing) Are they still just as far apart as they were or not?

D: No.

I: Okay, why not?

D: Because (hesitates) all there is is a fence between them. . . They're still far apart, they're not further . . . apart.

Debbie was the first student for whom "just as far" meant "farther." For Audrey, "just as many" meant "more." For 5 other students, "just as much" meant more (or perhaps $\geq$), as seen in the following example:



I: Are there any of these shapes that have just as much room inside as this blue one does?

C: A

I: Any others?

C: No.

I: How could you tell which one?

C: Cause it's bigger.

I: The question was, I wanted you to have <u>just as much</u> room inside as this one.

C: Uh huh.

I: This one has just as much room?

C: Yes.

I: I thought you just said it was bigger.

C: It still got much as room, don't it? more room.

I: You're saying more room, though. Can you find one that has the same amount of room?

C: The same amount?

I: Uh huh.

C: This one here. (E)

I: How could you tell?

C: They got kind of a box, they look the same.

Besides the five students who interpreted "just as much" as "more" or possibly "at least as much," there were two other students who seemed to either be transitional with respect to the meaning or to have two meanings. That is, they started out similarly, but were able to shift to the investigator's meaning.

These curiosities may be partly explained by studies discussed by Novillis (Note 10). In referring to 3 to 6-year-olds, she remarked that in Piagetian research, there is a need to check children's language acquisition. She mentioned that words like "larger" often acquire a positive connotation, and that for some children this connotation lingers. (It has been found that the word "more" is used eight times more often in conversation than "less.") She also reported that some third graders who had been doing subtraction for two years did not understand the meaning of the word "less."

How could "just as much" have meant "more" for these students? Maybe they considered it a good thing to have "just as much" candy as another; if "just as much" was good, then it may have acquired the positive connotation, "more." Perhaps it is an ethnic idiosyncrasy; seven of the nine students mentioned were Black.

On the other hand, in considering the understanding and use of expressions such as "same number," "just as many," etc., Flavell (1963) says that the crucial question is not whether vocabulary growth takes place, but whether anything else takes place, and what the relation is between the vocabulary growth and this something else. He believes that in most of Piaget's studies,

> whatever vocabulary change occurs is in large measure a consequence, reflection, or symptomatic expression of an underlying and more fundamental cognitive change. (p. 434)

The relationship of vocabulary development to cognitive structure is probably complex and variable, he continues, and the mastery of some words has much more important cognitive-developmental implications than the mastery of others (pp. 434-435).

Finding 4. Lack of experiences. It is possible that many of these students had not considered the ideas in the tasks before. Some examples where experience seemed to be lacking were given in earlier sections. There were students who had apparently never classified geometric objects by shape before. Some did not know the distinguishing properties of triangles, calling some of them rectangles. One student drew what he thought was a triangle; it was a rhombus.

Donnie could put some things in groups but still had difficulty sorting. Even after the example of grouping by color, he put the four large circles together, but named it "the red, blue, yellow, and green group." Things "going together" seemed to mean to him "used contiguously," rather than "having common characteristics" in the pictorial subtask. His groups were:

a. pieces of furniture, flowers, telephone, bread, pancakes (almost figural, as if he is making an arrangement), then a man.

b. man with javelin, sailboat, man in boat, Mickey Mouse

c. two men, watch, portrait

d. cars, suitcase, salad

I: What would you call each of these groups?

D: (b) Sports; (c) have fun time; (d) cars are like for traveling and going different places, and you have to have some kind of food, you know, if you're traveling or whatever, and I put the bag to put your clothes in or whatever.

(a) Things that you have in your house you got to have--food, furniture and stuff like that and a telephone.

I: So what's he doing here (Pointing to man)?

D: Hmm. . . He's driving and looking out the window.

I: Is he with that group?

D: Yes maam.

I: Why is he with that group?

D: Uh, . . . Can I change him?

I: Sure.

D puts man with group d.

I: Oh, you're going to put him going on the trip.

It is suggested that this classifying activity was a new experience for Donnie.

Even when he answered correctly, Wayne was apparently in unfamiliar territory. Some of his comments and asides on the tasks were:

> This is a wierd test.
>
> Is that how you want them?
>
> I know I goofed up that time. Did I goof up?
>
> What do they have this test for, crazy people? To prove that somebody's crazy?

On the concrete fractions test, where he was less successful, he said:

> (Asked to point out about where 3/6 would be on a string number line) I ain't studied no rope in my math before. I don't see no 6 on here, I don't even see a 3.
>
> (Asked to circle 1/3 of a group of chips) The school ain't showed me how to do it yet.

During the written test, he confided that he had gone to a private school, where students did not have to study mathematics if they did not want to. (This was not checked.)

The word "area" was not used by the investigator. It was spontaneously used by only one student, whose overall performance was one of the two best ones. Everett was also the first student who could answer correctly about the area of the construction paper rectangles but not of the geoboard. The other one was Jackie, whose overall performance was the other of the two best ones. It was as if these two had had some experiences with the concept of area, if not with the geoboard. In Jackie's protocol, given earlier, she referred to the paper squares as "blocks." Did she refer to blocks that a child plays with, perhaps a younger sibling? Or could she have meant concrete blocks used in building? Was there an actual physical referent, which made the extension to the pictorial representation work for her? In a less structured interview, she might have been asked,

What does this shape remind you of? or

Why do you call them blocks?

Many students seemed to enjoy manipulating objects, making designs or experimenting. Such experimentation was often cut off in the process of moving to the next task.

As has been indicated, language development was weak even in places where numerical concepts seemed to be present. Perhaps these students, many of whom seemed to be culturally deprived, had not discussed any intellectual concepts in their homes, especially concepts like area. Perhaps they were not encouraged to play games which required mathematical reasoning, like chess or Monopoly. No information about these possibilities is available.

But it has been observed that in compensatory classes there is a sense of urgency about preparing for the literacy test, and about conquering the basic skills one by one; that neither school board nor principal nor teacher nor even students have any patience with anything not seen to be directly on task. Prevalent teaching strategies observed, lecture-recitation and individualized instruction, do not encourage peer discussion of mathematical ideas, even those related to the basic skills. If use of manipulable materials is seen as frivolous or a waste of time or childish, learning steps which may be necessary for some students are omitted.

Finding 6. Learning during assessment. It was noted that there seemed to be some overall advantage on the written fractions test for those who took the concrete fractions test first. This possibility is important if it indicates that learning was occurring during the administration of the concrete test. It seemed necessary to investigate this possibility in more detail.

Some of the pairs of individual items were analogous; two such pairs will be considered.

Concrete version	Written version
A-1. Arrange 3 blue chips and 5 white chips in a row. Say: "What fraction of the chips are blue?"	A-1. Circle the fraction this picture shows:  3/4 3/7 1/2 5/3

Scores were as follows:

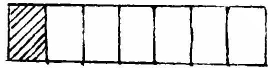
Concrete first, 5 of 12, or 42% correct	Concrete first, 7 of 12, or 58% correct
Written first, 3 of 13, or 23% correct	Written first, 4 of 13, or 31% correct

C-1. Arrange rectangle with 3 red and 2 yellow small rectangles.

Say: "What fraction of the large rectangle is red?"

Concrete first, 6 of 12, or 50% correct
Written first, 10 of 13, or 77% correct

C-1. What fraction of the rectangle is red?



Concrete first, 8 of 12, or 67% correct
Written first, 4 of 13, or 31% correct

These two item pairs are typical. On the left we see no trend; first one group is favored, then the other, on the concrete test. But the pattern of the students who took the concrete test first being more successful on the written test continues throughout the two sections where objects were manipulated, sections A and C. No claims are made of statistical significance; the inference was made, however, that the concrete test protocols should be examined more closely for evidences of learning.

In fact, in reviewing Table 7, the tabulated scores for the concrete fractions test, one might notice a trend of improvement within section A. That is, a glance shows that the number of Passes are more numerous in the bottom half of the section (in contrast to sections B and C, for example, where the later items seem to be harder). As mentioned earlier, the instrument was designed to have easier questions first, and to increase in difficulty within sections.

Apparently this was not accomplished. Are the items easier in the second half of the section, or are the students learning, even from the ones they missed?

Individual student scores can be examined in the vertical columns. The fourth student, Martha, and the last one, Ronnie, share a pattern of Questionable scores followed by Passing scores. Debbie, sixth from last, missed the first three questions entirely before receiving four Passes. The items were reexamined.

The first three items in section A were:

1. Arrange 3 blue chips and 5 white chips in a row. Ask: "What fraction of the chips are blue?"

2. Make a 2 by 3 array, 2 blue in column. Ask: "What fraction of the chips are blue now?"

3. Rearrange array; put 3 blue, but alternating, not next to each other, and 3 white. Ask: "Now what fraction of the chips are blue?"

Here it was thought that the student only had to recognize or identify, not construct or synthesize.

In items 4 through 7, the student was asked to perform and demonstrate (more advanced behaviors):

4. Make an array of 10 blue chips. Give subject string knotted together at the ends. Say: "Can you put this string around 3/10 of the chips?"

5. Make available chips of both colors, also string. Say, "Can you use the chips to show what is meant by the fraction 2/5?"

6. Make chips and string available. Say: "Can you show what is meant by 5/5?"

7. "Can you show what is meant by o/5?"

Analysis reveals that, for each of items 1, 2, and 3, the student must look at the group of chips and decide what is the whole and what is the part. Students who do not have the part-whole concept firmly fixed may answer 3/5 for item 1, 2/4 for item 2, and 3/3 for item 3.

But in item 4, the denominator, the whole, is made clear to the student, for the directions say to put the string around 3/10 of the chips. There were 10 chips there, and the investigator had called attention to 10; 64% of the students correctly put the string around 3 chips. Three students who had missed the first three items were "forced" by the directions to succeed with item 4; they then proceeded to answer item 5 correctly. Debbie answered all the rest of the items in the section correctly, even the troublesome exercises involving 0 and 1. Here are her responses:

(A-6):

I: Can you show what is meant by 5/5?

D: (puts out a group of 5 blue chips) All blue chips.

I: So all of them. . .

D: are 5/5.

(A-7):

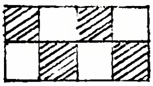
I: Can you show what is meant by 0/5?

D: (Frowns, bites lip) No chips at all?

I: So what would you do?

(D pushes away the chips.)

Daniel had also started off poorly and changed his pattern at A-4. Then, on the area concept of fraction, he answered C-1 and C-2 correctly. C-3 was presented:



Asked what fraction of this rectangle was red, Daniel said 4/4.

On C-4, he correctly used one red and one yellow rectangle to show what was meant by 1/2; then he asked permission to go back to C-3. He changed his answer to 4/8. (He was not given any indication, or course, of whether he was right or wrong either time; he was producing his own feedback.)

All but one of the five students who seemed to be learning on the concrete test had already taken the written test. On examination of his written test, it was found that Leon's freshly acquired concept of fraction had apparently persisted through sections A and C of the written test. He answered 7 of the 9 items correctly, and the two he missed were difficult ones, answered by only 1 and 3 students. Leon had received no instruction or feedback.

<u>Finding 1. Students' deficiencies</u>. The findings discussed above have illustrated some of the deficiencies referred to in the statement of this finding. There were other results which seem worthy of mention whether any interpretation is available or not. The first group of observations comes from the tasks.

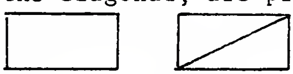
In the conservation of number (I), subtask A-3, students are shown a set of 20 chips and asked, "Can you divide these chips up into four equal sets?" One sample protocol has already been given. This task was performed correctly by 8 students. Five sets of four chips were formed by 17 students; upon being questioned, 13 saw their mistake and were able to correct their solutions. But 4 were still not successful; 3 of them maintained that it could not be done. Apparently the number "4" used in the directions was a strong signal for them; for some students it seemed to block other approaches to the problem.

In the pictorial form of that task, there was the possibility of miscounting or adding wrong. Even though there was no time limit and pressure was minimized, 18 spots were variously miscounted as 12, 15, and 24, and one student said that 9 and 9 are 12. Both types of error would yield wrong answers to the concept question, masking whether the student actually had acquired the concept or not. These errors were not corrected by the investigator.

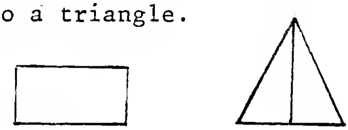
In Chapter 2, the behavior of children who were transitional was described; for them, logic sometimes prevailed, sometimes perception. On the farm area task (VI, A-1), students who received Passing scores answered correctly through at least four barns and seemed very confident of their answers, even saying things like, "You can put barns there all day, and it will not make a difference." It was discovered later that Piaget's colleagues, in their patience, often continued the questioning up to 15 or 20 houses. They cite a case in which a student answered correctly up through 14 houses, but changed his mind at the 15th house (Piaget, Inhelder, & Szeminska, 1960, p. 263). It is not known whether the students in this study would have changed their minds in the face of a greater perceptual influence.

The following protocol does give a vivid example of logic giving way to perception:

In VI, B-2, two construction paper rectangles, one cut along the diagonal, are presented. It is established



in the first question that the two rectangles take up the same amount of space. Then one of the rectangles is rearranged into a triangle.



I: Do these two shapes take up the same amount of room?

M: Uh huh.

I: How do you know?

M: You just cut part of yours, made it a different way, but they got the same amount. . . They the same size.

I: But you would say they take up just as much room, just as much space.

M: (Looks, frowns, squints) No, I'd say yours got more. . . more room.

I: This one?

M: Yeah, yours is taller.

Mike not only answered the question correctly, he justified his answer; then he looked at the shapes and perception became the stronger influence.

Some insights were revealed by the interview process which might have been missed otherwise. In task VI, A-2a, shown earlier, students were shown a geoboard. A 1 by 3 rectangle, the standard, and five other shapes were shown. Students were asked,"Is there a shape here that has just as much room inside as this blue one?" There were 8 students who selected the two shapes which contained three units of area. In a conventional situation, the test scorer might have thought the students had attained the concept of area, or at least of "room inside." However, when asked, "How did you know which ones to pick?" 7 of the students revealed that they were counting the pegs which held the rubber bands, rather than the units of area. This method gives the correct result sometimes, but it does not exemplify the concept of area.

A summary of their strategies on this subtask were: comparing shapes ("just looks like it," etc.), 13 students; counting pegs, 9; combination of the two, 1. In combination with peg counting, 2 students used the flat of their thumbs as measuring devices, placing them inside the shapes in the only real attempts to consider area on this subtask. One of these arrived at the correct answer and was given a Questionable score (because of the peg counting); he was unsuccessful on the remainder of the area subtasks.

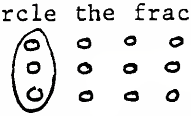
Written fractions test. It has been shown that students often responded on the concrete test with a part-part, rather than part-whole, concept of fraction. A part-part response was always a good distractor on the written test also, as these examples show.

A-1. Circle the fraction this picture shows:

$\frac{3}{4}$ $\frac{3}{7}$ $\frac{1}{2}$ $\frac{5}{3}$

The correct answer was chosen by 44%, 4% skipped the item, and the rest chose 3/4, the part-part response.

A-2. Circle the fraction this picture shows:



$\frac{3}{9}$ $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{3}$

This item was answered correctly by 12%; another 12% chose 1/3; and 76% circled 3/9.

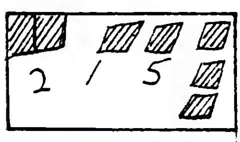
No particular pattern is evident on the number line model of fraction. Here is an example. (Beneath each answer choice is the percentage of students who chose it.)

B-1. Circle the answer. Which fraction names the distance from O to Δ?

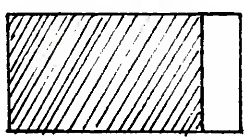
3	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{1}{2}$
16%	16%	4%	36%	28%

The distribution of results probably indicates random guessing; certainly those students who chose 3 as the answer can have no understanding of this meaning of fraction.

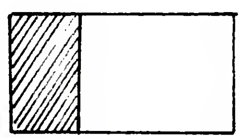
Two questions on the written test which caused difficulties in scoring were C-4 and C-5, in which students were asked to shade 1/3 and 2/5 of a rectangle, respectively. Some students' attempts to shade 2/5 of a rectangle are shown below:



Example 1



Example 2



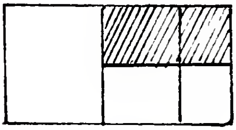
Example 3

Example shows 2 small pieces and 5 small pieces, the familiar part-part concept; the student has even labelled the parts.

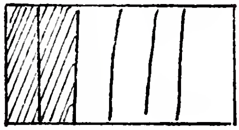
In the second example, the student understood that a portion of the rectangle was to be shaded, but did not have a correct notion of the relative size of 2/5. There were 5, or 20%, of the responses which looked like this example. There were two procedures used; some students drew the vertical line first, and then shaded in up to the line. Others just started shading; when they thought they had shaded enough, they stopped, drew a vertical line, and tidied up the shading. The same two procedures produced other distinct groups of results. Three students (12%) had almost identical diagrams--about 1/4 of the rectangle was shaded. Another student shaded slightly over 1/2 of the rectangle. Three students each made a vertical mark to divide the rectangle in approximately the right proportion, but did not shade either region. The only one (4%) which was hesitantly scored as correct was Example 3 above, which is actually about 1/3 of the rectangle (fairly close to 2/5 if an estimate). As in all written tests, however, it is not known whether the student was actually estimating or had made a luckier guess than the others. On the previous question, asked to shade 1/3, she had shaded about 2/3. The investigator's note says:

> Just starts shading from left; each time I think she is done, she shades a little more.

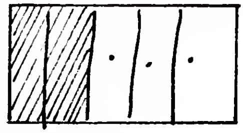
And on C-5, the one shown above, the note says, "No anticipatory scheme, just starts shading."



Example 4



Example 5

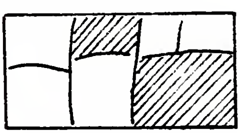


Example 6

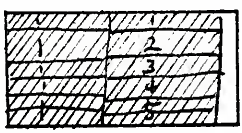
In Examples 4, 5, and 6, the students knew that partitioning was in order. Results like that shown in Example 4 were produced by 3 students (12%). They drew a vertical line dividing the rectangle in half, subdivided the halves, and then erased the lines within one of the halves. Thus they ended up with 5 pieces, though they were not equal, and shaded 2 of them.

Others partitioned the rectangle with vertical lines. One student made two vertical lines from the left end only, and shaded the two regions formed. Two students made 5 lines instead of 5 regions, subsequently shading 2 of 6 regions, as shown in Example 5. One student then recounted and erased one line, making 5 (unequal) regions (Example 6).

Two students skipped the question. The remaining three unique responses are given below. Example 7 shows the



Example 7



Example 8

Example 9

result of partitioning the rectangle into 7 unequal regions and shading 2 of them. Example 8 shows Virginia's second attempt; she had already erased a similar configuration. And Example 9 seems to be a guess.

It was mentioned that often even very young students seem to have an intuitive notion of one half. On the written test, 18 students, or 72%, could circle 1/2 of a row of 8 chips. But an interesting result was noted in the section on equivalent fractions. Here is item D-1:

Circle the fraction that has the same value as 1/2.

$\frac{1}{1} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{3}{6} \quad \frac{4}{10}$

While 6 students, or 24%, correctly chose 3/6, the best distractor was 1/1, which fooled 12 students (48%). Repeated pondering of this result produced the following possible explanation: Perhaps students were looking at the symbols figuratively, as if the 1/1 is a picture of something, cut into two equal parts. This is plausible, since the numbers were handwritten, as above. Mysteries like these from the written test cannot be solved without accompanying comments from the students.

Here is an example from the section on comparing fractions:

E-1. Circle the smallest: $\frac{1}{9} \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{6}$

The majority of the students (68%) chose as the smallest the fraction with the smallest denominator, even though that response was one half. If students had an intuitive meaning for one half, perhaps they were not connecting whatever notion they had with the symbols they saw on paper.

Similarly, in item E-2, 72% chose as the largest the fraction with the largest denominator, 1/10 (which was being compared with 1/8, 1/3, and 1/4). With the introduction of numerators other than 1 in the next two items, these error patterns were continued by 64% of the students.

No student could put in order, from the smallest to the largest, the following fractions, commonly used in measuring with inches: 3/4, 1/16, 1/2 and 3/8.

Finding 5. Motivation. There is a feeling prevalent in some quarters that students in compensatory classes are apathetic, not interested in learning. For the 25 students in the study, that was not the case. They were offered no reward for participating. Novelty, of course, could be a very motivating force if they were bored in their class, as could the fact that they would get out of class for two hours. It would be unfortunate if either of these represented very strong reinforcements. It should be noted here that all three of the teachers of these classes were friendly, dedicated individuals who seemed to be sincere in wanting to help the students; any avoidance response in the students would seem to be related to the sameness of the teaching strategies and materials, and/or to frustration due to lack of success, rather than to the teacher.

The students owed no loyalty to the investigator, who had only talked to them briefly over a period of a few days. The chance to spend an hour individually in the company of an adult, discussing mathematical ideas, might have been reinforcing. Or perhaps they had the altruistic motive of helping future generations of mathematics students by participating in the study.

For whatever reason, the students were all motivated; all wanted to please, to cooperate. One student, a senior, had already completed all her work for graduation, including passing her basic skills tests, and came back to school an extra day specifically to finish the experimental tasks. She did this in spite of the fact that she had not been doing well on the problems and may have been experiencing some cognitive conflict.

Many of the students were interested in receiving feedback, which was not available until after their testing was completed; a few stayed over into lunchtime to find out the correct answers to some of the questions. Upon finding out that their work was not correct, no student blamed anyone else. One student said, "I'll have to work a little harder on fractions next year."

Considerable motivation to learn was exhibited by those students who, though not understanding the concept of fraction, managed to memorize algorithms for the addition and subtraction of fractions.

Limitations of the Study

The limitations of the study can be divided into two types. The first type has to do with the design and concerns the possible relationships proposed in Questions 1 and 2. The other type, involving individual parts of the instruments, concerns primarily the qualitative data.

Possible relationships, Questions 1 and 2. First, the Walbesser contingency table is suited for heterogeneous groups, so that random samples, with no existing relationships between behaviors, would have random frequencies in all four cells. In this study many of the students could perform neither of the two behaviors being examined; consequently there was a large number in one cell, very little elsewhere. For those cases, this method of data examination yielded no information. It was not known prior to the study that this would be the case.

Secondly, to objectively compare the two modes of presentation on the tasks, concrete and pictorial, the two modes should not have always been administered in the same sequence. Both sequences should have been used, as was done with the fractions tests.

Qualitative data. Defects in even the revised instruments were mentioned in some detail in previous sections. Sometimes they meant that the instrument did not test what was intended. Sometimes they confounded the analyses of the possible relationships mentioned above. These will be listed here without repeating the discussion.

In the conservation of number and the seriation tasks, the two modes of presentation, concrete and pictorial, were not analogous.

In the classification task, the geometric objects used in the concrete formulation seemed unfamiliar to many of the students.

In the class inclusion task, the two modes did not seem too different, but the numbers of objects being considered were different. In the concrete version, 5 of 8 objects were used, compared to 8 of 11 pictures in the pictorial version.

The geoboard, used in the conservation of area task, was not familiar to most of the students.

The written test, given as it was without questioning, did not yield as much information as it might have if accompanied by an interview ascertaining how the students arrived at their answers.

In the next chapter, some implications and questions for research and for teaching will be considered.

CHAPTER SIX
IMPLICATIONS FOR RESEARCH AND FOR TEACHING

A major finding of this study is the depth of the deficiencies of some secondary school students in certain areas of cognitive development and in certain concepts of arithmetic. Even more serious is the likelihood, implied by the casual way in which the subjects were chosen, that these 25 students were not unique among Florida secondary school students.

Other more optimistic findings indicate that there exists reason to believe that this situation can be changed, and that a greater percentage of successes might be produced among students in compensatory mathematics classes.

In consideration of the main questions proposed by this study, no clear relationships were established. Nevertheless, the information gained indicates that this line of research may prove to be fruitful in the future.

Several specific questions uncovered in the study of the protocols seem to be worthy of further research.

This study was not concerned with instruction. But insights gained about diagnosis may be of help to teachers and those responsible for assessment.

These implications will be discussed in two parts--first, the questions for future research.

Implications for Future Research

The depth of deficiencies that exist in some students' thinking was revealed by the study. Still, the thinking of students is not clearly understood. For example, in the task in which students were to divide a set of 20 chips into 4 equal subsets, the question arises as to why this was so difficult for a few students. As mentioned earlier, 3 students made sets of 4 but said that 4 equal sets could not be made. Since such partitioning seems logically necessary for understanding of a discrete model of fraction, students' performance on this task might be investigated with sets and subsets of various sizes.

The methods used may be of value in examining what processes are being used by students in other areas of mathematics learning. The concept of fraction was chosen for this study because it was a key concept on which was based the study of decimals and percentage and of further mathematics courses, and also because the lack of understanding of fractions often seemed to mark the division between those who were successful in mathematics and those who were not. There are many other segments of content which would lend themselves to intensive study. Possible examples are the numeration system (whole number and/or decimal), the measurement of length (including estimation), compound multiplication or division, and the writing of number sentences.

In addition, the topics in this study could be studied more intensively, perhaps with fewer students and less structured interviews which could explore more fully the students' mental processes.

There is reason to believe that there is hope that students like those in the study can continue their development. Piaget's theory does not preclude further development at any age; and many of these students neatly fit the descriptions of transitional students. Some training studies seem to indicate that perhaps Piagetian concepts can be taught; this seems especially probable in the case of transitional students. There were, both in the literature and in the protocols of this study, instances where learning seemed to be occurring during assessment. The possiblity was mentioned that transitional students, confronted with problems they wished to solve, were being provided with the cognitive disequilibrium Piaget says is necessary for growth.

Specifically, there seemed to be some learning on the concrete fractions test which transferred to the written form of the test. Whether this can be induced experimentally seems worthy of study. Furthermore, if this finding can be substantiated, it must be asked which aspects of the concrete test were effective in producing this learning. Was it the objects themselves, or the fact that the student was manipulating the objects? That is, would the same effect have been achieved by having the students watch the objects being manipulated? Was the learning effect due to the open-ended questions, which produced conflict, interactint with the concrete materials the students could use to work out the conflict?

Question 1

Steffe (1968) raised the question: is it possible to devise a readiness test for fractions, possibly using the psychological theory of Piaget? The answer is a tentative Yes. There were difficulties with the instruments; for the relationships examined, either no information was gained, or the proposed relationship was contradicted by, at most, a frequency of 1. These relationships might still be found in a study which provided for limitations present in this study. There may be other underlying cognitive processes that need to be studied. Students need a part-whole concept for the understanding of the concept of fraction, for example. But what is necessary for the understanding of 0 as 0/5, say? Or of 1 as 5/5? This exploratory study might be the first step in the process of developing a readiness test for fractions.

Question 2

Due to defects in the instrument, no statement can be made about the relationships between the concrete and pictorial formulations of the individual tasks used. As indicated, there did seem to be an advantage on the written fractions test for those who took the concrete fractions test first. This needs to be investigated further. It is suggested that such a written test should also be accompanied by requests to the student to "tell how you solved that" or other questions to expose the processes used.

Other Issues

The behaviorists had convinced many educators that students needed feedback early and often for learning to take place. This position is questioned, and deserves further research.

Even if the needed underlying cognitive structures of the concept of fraction (or other mathematical content) are identified, are these structures to be taught? If so, how?

How can assessment of true understanding, rather than just rote learning, be made on a large scale?

Vocabulary meanings need to be investigated further. Are there other words or phrases, crucial to communication of meaning in mathematics, which are not mutually understood by students and teacher? How does mathematics vocabulary develop? Research on the relationship of vocabulary development to cognitive structures is needed.

Suggested Research

Some of the above issues can be confronted experimentally and some through a clinical method. Many of the basic questions of learning can be researched in teaching experiments.

A possible teaching experiment to consider Question 1 further would involve as content the fraction concept and underlying cognitive structures thought to be related. A heterogeneous group of students, preferably younger ones, to avoid the confusions of rote algorithms, could be the subjects. The experiment would begin with intense and thorough diagnosis. Careful notes could be kept of interviews at various stages throughout the instruction. The students themselves could be asked to report strategies that help them to learn. A variety of teaching strategies could be used; some that probably should be included are:

--manipulable materials

--interviews and/or opportunities for students to work cooperatively, to force discussion of ideas

--open-ended questions and experiments

--intermittent feedback

To analyze the results, the procedure of Brainerd (1979) might be used. He describes a simple (in concept, at least) way to test whether the "readiness effect" exists. His remarks were made with reference to training studies, but the method seems appropriate to a teaching experiment. He suggests that in a training experiment, the learning curve, from pretest to posttest, is roughly linear. "The readiness view predicts that the slope of the regression line, which is the sum of two functions, must be steeper for experimentals than controls" (p. 11). It is presumed that if the readiness attribute (in this case, the acquisition of the Piagetian developmental structures) has no effect, then the learning curves will be parallel, though with different intercepts depending on prior knowledge as exhibited on the pretest. If "readiness" does exist and can be identified, the slope will be steeper for those students who were "ready" at the beginning of the experiment.

Implications for Teaching

Although this study was concerned with diagnosis and not instruction, there are implications for teaching in the qualitative findings given in Chapter 5. Their importance to instructors and to persons responsible for planning instruction are discussed below.

1. Many of the students, even graduating seniors, exhibited severe deficiencies both in Piagetian concepts and in the concept of fraction.

This fact can, perhaps, be thought of as an indictment of past instructional strategies, which have not produced the desired learning in certain secondary school students. It also casts doubt on the validity of assessment procedures through which these students have been "passed," accredited as having mastered basic skills, and in some cases received high school diplomas. These possible implications depend on instructional planners' agreement that the concepts tested in this study are concepts that are desirable as educational outcomes for all students. If such agreement exists, then some reexamination of both instructional strategies and assessment procedures are in order.

Specific implications for teachers working with students follow, in discussion of the other findings. Findings 2 and 3 are discussed together.

2. Some students could perform rote algorithms without understanding.

3. In spite of precautions, vocabulary was a problem; some students have non-standard meanings for some words and phrases.

Teachers can not assume that a student who passes a 4-question basic skills test on adding fractions actually has a concept of fraction. And the vocabulary difficulties uncovered imply that teachers can not assume that information given to students is actually assimilated by students in the form in which it was given.

A logical inference seems to be that some evaluation other than written tests is needed. On the strength of previous experience and the findings of this study, the investigator recommends that some assessment strategy be found which includes oral feedback from the student. Such a strategy need not be formal, and practical considerations would prohibit the use of this format on every testing occasion. But it seems obvious that the student should be provided some opportunity to verbalize information that is being received, so that the teacher can see whether the student receives the same information that is being transmitted.

Interaction with other students, in discussion of mathematical topics, may also assist the student in developing both mathematical vocabulary and concepts.

4. Some students have apparently not had the experiences necessary for development of the mathematical or Piagetian concepts.

It is not possible for a teacher to know the extent of the experiences in each student's background. To ensure maximum success for the greatest number of students, it seems reasonable that a teacher should provide different models of concepts being taught, with different materials, specifically manipulable materials wherever possible.

Opportunities to classify, to look for common attributes of objects, should be provided.

Experience with the number line model of fraction seems to be indicated, before any attempt to have the student measure length.

And, as mentioned above, students should be called upon to express their thinking about mathematical concepts, both to the teacher and in interaction with other students.

It would be discouraging to report findings 1, 2, 3, and 4 without findings 5 and 6, which provide a basis for optimism:

5. Students were cooperative, interested, and motivated to learn and to succeed.

6. Learning may have been occurring during the testing sessions.

There is a belief (more prevalent in some circles than others) that students in compensatory and remedial programs are apathetic, disinterested, even hostile. The experienced teacher is often "rewarded" by not being assigned compensatory classes, which are therefore the job of the least experienced teachers, sometimes of the teachers with little or no mathematics background, who are teaching "out of field."

The students in this study, dealt with as described in the Procedures section, disproved that belief. They uniformly responded in a cooperative manner to the opportunity to discuss mathematical concepts with an adult. A teacher who arranges occasional times for individual discussion with students may not only learn more about their mathematical knowledge but also more about ways to help them increase that knowledge.

Finding 6 implies that there is hope. These students have not reached a ceiling on their development, or a dead end. If students can learn within 2 hours of testing, surely within a school term they can accomplish much. It behooves researchers and teachers themselves, in action research in classrooms, to look for, try out, and record the results of different teaching strategies to help effect this learning. Potentially successful strategies include the use of manipulable materials, active experience with fraction concepts in various forms, and small group work in which student discussion of mathematical ideas can occur.

APPENDIX A
PIAGET-TYPE TASKS

I. Conservation of Number--Concrete

1. Give one of us 8 chips, the other 10 chips.

   Ask: CAN YOU FIX THESE SO THAT YOU HAVE JUST AS MANY AS I DO?

2. Say: NOW WE HAVE THE SAME NUMBER OF CHIPS.

   Rearrange one row of chips into a circle. Say: NOW WHO HAS MORE CHIPS, YOU OR ME, OR ARE THEY THE SAME?

   WHY?

3. Spread out one row of chips into a long line. Say: WHO HAS MORE CHIPS, YOU OR ME OR ARE THEY THE SAME?

4. Put 16 chips out. Say: CAN YOU DIVIDE THESE CHIPS UP INTO FOUR EQUAL SETS?

   Spread out two of the sets of four. Ask: ARE THERE MORE HERE OR MORE HERE?

5. Arrange 12 black chips in 3 by 4 array and 12 blue chips in two 2 by 3 arrays. Ask:

   ARE THERE MORE BLACK CHIPS OR MORE BLUE CHIPS OR ARE THEY THE SAME?

II. Conservation of Number--Representational

1. Present picture of 6 red chips in a row, and 6 blue chips in a row, but with the blue chips spread out more.

   Say: ARE THERE MORE BLUE CHIPS OR MORE RED CHIPS OR ARE THEY THE SAME?

2. Present picture of 12 blue chips in a 3 by 4 array, and 12 red chips in two 2 by 3 arrays.

   Say: ARE THERE MORE BLUE CHIPS OR MORE RED CHIPS OR ARE THEY THE SAME?

III. Seriation-Concrete

1. Present 10 drinking straws, ranging in length from about 10 to 25 cm.

   Say: CAN YOU PUT THESE IN ORDER?

2. Present card, with five of the straws arranged in descending order. Present other five straws of intermediate lengths (between the lengths of longest one and the shortest one on the card).

   Say: SOME OF THESE HAVE ALREADY BEEN PUT IN ORDER. CAN YOU PUT THESE WHERE THEY GO?

IV. Seriation-Representational

Present diagram with straws arranged in order by length, the spaces between them marked by letters a through j. On the right is another straw.

Say: IF YOU WERE GOING TO PUT THIS STRAW IN ORDER WITH THE OTHERS, WHERE WOULD IT GO?

WHAT LETTER IS THAT?

V. Classification-Concrete

Present collection of plastic circles and triangles in two sizes and 4 colors. Group them by color.

Say: I HAVE THESE GROUPED ACCORDING TO COLOR. CAN YOU ARRANGE THEM ANOTHER WAY SO THAT THEY GO TOGETHER?

CAN YOU GROUP THEM ANOTHER WAY?

ANOTHER WAY?

VI. Classification-Representational

Present pictures out of magazines (automobiles, food, furniture, men, and a few oddball items: a watch, a telephone, Mickey Mouse, a suitcase, a portrait of an eighteenth century gentleman).

Say: CAN YOU GROUP THESE IN SOME WAY SO THAT THEY GO TOGETHER?

VII. Class Inclusion-Concrete

Present geometric shapes, all red, but different shapes, thicknesses, and materials.

Say: HERE ARE SOME RED SHAPES. CAN YOU GROUP THESE SO THAT THEY ARE ALIKE IN SOME WAY?

Ask the name of each group. Say, for example:

WHAT WOULD YOU CALL THIS GROUP?

Say: ARE THERE MORE RED SHAPES OR MORE CIRCLES?

ARE THERE MORE RED SHAPES OR MORE LARGE SHAPES?

IF I TOOK ALL THE TRIANGLES AWAY, WOULD THERE BE ANY RED SHAPES LEFT?

IF I TOOK ALL THE RED SHAPES AWAY, WOULD THERE BE ANY CIRCLES LEFT?

VIII. Class Inclusion--Representational

Present pictures of food (fruits, meats, crackers, cakes, etc.). After examination, ask:

CAN YOU THINK OF A NAME TO CALL ALL THESE PICTURES?

(Answer: food)

CAN YOU MAKE SOME GROUPS OF FOODS THAT ARE ALIKE?

WHAT WOULD YOU CALL THESE GROUPS OF FOOD?

IS THERE MORE FOOD OR MORE FRUIT?

IF I TOOK AWAY ALL THE FOOD WOULD THERE BE ANY FRUIT LEFT?

IX. Conservation of Distance--Concrete

Present two toy men, about 2 to 3 inches in height. Place them on the table about 20 inches apart.

Ask: ARE THESE MEN CLOSE TOGETHER OR FAR APART?

Place a low screen, or barrier, like a ruler, for example, between the two men.

Ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

Place a cardboard screen higher than the toys between them.

Ask: HOW ABOUT NOW? ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

Place a block of wood about 3 inches wide between the toys.

Ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

X. Conservation of Distance-Representational

Present drawing of two stick figures about 10 inches apart. Ask:

ARE THESE TWO PEOPLE CLOSE TOGETHER OR FAR PART?

Then place short "fence" (shorter than the figures) made of construction paper between the two figures. Ask:

ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

Place taller "fence" made of construction paper (taller than the figures) between the two figures.

Ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

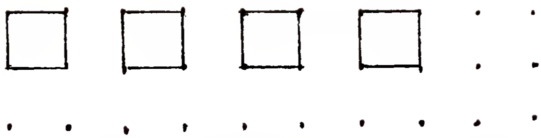
Then place a wide obstruction of construction paper (about 3 inches wide) between the two figures.

Ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

XI. Conservation of Area-Concrete

1. Present geoboard and rubber bands. Four rubber bands have been used to enclose four individual square units in a column.

   Ask: CAN YOU USE THE RUBBER BANDS TO MAKE A SHAPE THAT HAS JUST AS MUCH SPACE INSIDE AS THESE FOUR SHAPES?



*Present the same arrangement of four square units as before. Use one rubber band to enclose one more square unit not in line with the others.

Ask: CAN YOU USE THE RUBBER BANDS TO MAKE A SHAPE THAT HAS JUST AS MUCH SPACE INSIDE AS THESE?

2. Present two rectangles made of construction paper squares.

Ask: ARE THESE TWO RECTANGLES THE SAME SIZE? DO THEY HAVE THE SAME AMOUNT OF SPACE INSIDE? DO THEY TAKE UP THE SAME AMOUNT OF ROOM?

Rearrange one of the squares in one of the rectangles.

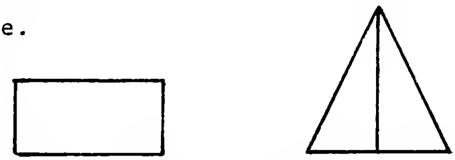
Ask: WHAT ABOUT NOW? DO THEY TAKE UP THE SAME AMOUNT OF SPACE (ROOM)?

3. Present two construction-paper rectangles, one of which is cut along the diagonal to form two triangles.

* Added at the first administration.

Ask: DO THESE TWO RECTANGLES TAKE UP THE SAME AMOUNT OF SPACE (ROOM)?

Rearrange one of the rectangles into an isosceles triangle.



Ask: DO THESE TWO SHAPES TAKE UP THE SAME AMOUNT OF SPACE (ROOM)?

XII. Conservation of Area--Representational

Present two construction paper "farms."

Put a "cow" on each farm.

Ask: DO BOTH OF THESE COWS SEEM TO HAVE THE SAME AMOUNT OF GRASS TO EAT?

Put "barn" on each farm. Ask: DOES THIS COW STILL HAVE THE SAME AMOUNT OF GRASS TO EAT AS THIS ONE?

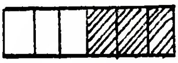
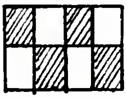
Put additional barn on each farm. On one farm, put the barn right next to the other one; on the other farm, put the barn some distance away from the first.

Ask: NOW DO BOTH COWS HAVE THE SAME AMOUNT OF GRASS TO EAT?

Continue adding barns, juxtaposed on the one farm, and spread out on the other, asking each time:

NOW DO BOTH COWS HAVE THE SAME AMOUNT OF GRASS TO EAT (AS EACH OTHER)?

APPENDIX B
CONCRETE FRACTIONS TEST

The investigator will do:	will say:
I. The concept of fractions	
(Area model)	
1. Arrange small yellow and red rectangles into rectangle, with 4 yellow and one red. 	1. I'M MAKING ONE LARGE RECTANGLE FROM THESE SMALL RED AND YELLOW RECTANGLES. WHAT FRACTION OF THE LARGE RECTANGLE IS RED?
2. Arrange 3 red and 3 yellow. 	2. NOW WHAT FRACTION OF THIS LARGE RECTANGLE IS RED?
3. Arrange 4 red and 4 yellow in an alternating fashion. 	3. WHAT FRACTION OF THIS RECTANGLE IS RED?
4. Put out several red and yellow small rectangles in no order.	4. USING AS MANY OF THESE AS YOU WISH, ARRANGE THEM TO SHOW WHAT IS MEANT BY THE FRACTION ONE HALF. (If necessary, ask: WHICH COLOR IS 1/2?)

5. Make both colors available.	5. NOW ARRANGE SOME OF THEM TO SHOW WHAT IS MEANT BY THE FRACTION 2/7.
6. Make both colors available.	6. CAN YOU SHOW WHAT IS MEANT BY 3/3?
II. <u>Concept of fractions (Discrete model)</u>	
7. Arrange 3 blue chips and 5 white chips in a row.	7. WHAT FRACTION OF THE CHIPS ARE BLUE?
8. Make an array of 6 chips, 2 blue.	8. WHAT FRACTION OF THE CHIPS ARE BLUE NOW?
9. Rearrange array; put 3 blue, but alternating, not next to each other, and 3 while.	9. NOW WHAT FRACTION OF THE CHIPS ARE BLUE?
10. Make an array of 12 blue chips. Give the subject a string knotted together at ends.	10. CAN YOU PUT THIS STRING AROUND 1/4 OF THE CHIPS?

11. Make avaliable chips of both colors and string.	11. CAN YOU USE THESE THINGS TO SHOW WHAT IS MEANT BY THE FRACTION 2/5? (If necessary, hint: CAN YOU USE THE BLUE CHIPS AND THE WHITE CHIPS? and/or CAN YOU USE THE STRING?) (If necessary, ask: WHICH IS 2/5?)
12. Make available chips and string.	12. CAN YOU SHOW WHAT IS MEANT BY 5/5?
13. Make available chips and string.	13. CAN YOU SHOW WHAT IS MEANT BY ZERO FOURTHS?
III. <u>Concept of fraction (Number line model)</u>	
14. Show string mounted on cardboard with fifths marked off with black magic marker, triangle pointing to the third mark.	14. WHAT FRACTION NAMES THE DISTANCE FROM ZERO TO THE TRIANGLE?

15. Show string marked off in eighths, mounted on cardboard, with letters of the alphabet at each mark.	15. WHICH LETTER IS AT ONE HALF?
16. Show model from 15.	16. WHICH LETTER IS AT 3/4?
17. Show string mounted on cardboard, with only the zero and one marked with black magic marker.	17. PUT YOUR PENCIL POINT ABOUT WHERE 1/4 SHOULD BE ON THIS NUMBER LINE.
18. Show model from 17.	18. PUT YOUR PENCIL POINT ABOUT WHERE 3/6 SHOULD BE ON THIS NUMBER LINE.

IV. Addition and subtraction of fractions

19. Cover bottom of box with cardboard rectangles to fit; 2 green, 3 beige.	19. WHAT FRACTION OF THE BOTTOM IS COVERED WITH GREEN?

20. On table, next to box: make rectangle of similar cardboard pieces, with 2 blue, 3 biege.

20. I HAVE MADE HERE A RECTANGLE AS BIG AS THE BOTTOM OF THE BOX. WHAT FRACTION OF THIS RECTANGLE IS BLUE?

21. Put the two blue ones in the bottom of the box with the green ones, taking 2 beige ones out.

21. WHAT FRACTION OF THE BOTTOM IS COVERED WITH GREEN AND BLUE?

22. Put one more blue one in.

22. WHAT FRACTION OF THE BOTTOM IS COVERED WITH GREEN AND BLUE NOW?

23. Take out the green and blue ones, and put beige back in.

23. WHAT FRACTION OF THE BOTTOM IS GREEN NOW?

24. Put 2 blue pieces back in, with 3 biege.

24. WHAT FRACTION OF THE BOTTOM IS COVERED WITH BLUE?

Take one blue piece out.

NOW WHAT FRACTION OF THE BOTTOM IS BLUE?

V. Comparing Fractions

25. Arrange two rectangles of construction paper, one purple and one green, to make one large rectangle, one fourth purple. Next to it, arrange another rectangle, with the amount of purple and green different; this time the purple and green sections are cut into one inch squares, more purple than 1/4 in investigator's rectangles.

25. IS THE FRACTION OF PURPLE IN THIS RECTANGLE LESS THAN YOURS, MORE THAN YOUR, OR THE SAME?

26. Provide more one-inch squares of both colors.

26. CAN YOU FIX THE SMALL SQUARES TO MAKE A RECTANGLE THAT HAS THE SAME FRACTION OF PURPLE AS MINE?

27.

27. WHAT FRACTION OF THE LARGE RECTANGLE IS PURPLE?

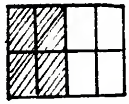
VI. Equivalent fractions

28. Make array of 12 chips.	28. PUT THE STRING AROUND 9/12 OF THE CHIPS.
29. Mess up chips and arrange into array of 12 again, 3 by 4 or 4 by 3.	29. NOW PUT THE STRING AROUND 2/3 OF THE CHIPS.

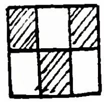
APPENDIX C
WRITTEN FRACTIONS TEST

1. What fraction of the rectangle is red? ______

2. The fraction that names the green part of the rectangle is ______.

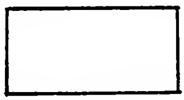


3. The fraction that names the yellow part of the rectangle is ______.



4. Color in 1/3 of this rectangle.

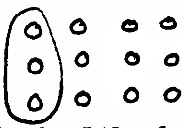
5. Color in 2/5 of this rectangle.



6. Circle the fraction this picture shows:

   3/4 3/7 1/2 5/3

7. Circle the fraction which this picture shows:



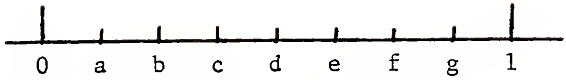
   3/9 1/2 1/4 1/3

8. Circle 5/9 of the chips:

9. Circle 1/4 of the chips:

10. Circle the answer. Which fraction names the distance from O to Δ ?

    3 1/3 3/5 2/5 1/2

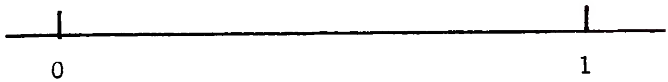
11. 

Which letter is at 1/2? ______

Which letter is at 1/4? ______

12. On the line, put a dot about where 1/3 should be.

13. On the line, put a dot about where 4/8 should be.



14. 2/5 + 1/5 =

15. 3/7
+ 2/7

16. 1/9 + 1/9 =

17. 1/4
+ 1/4

18. 1/2
+ 1/2

19. 3/7 - 1/7 =

20. 7/8
- 5/8

21. 2/5 - 1/5 =

22. 1/3
- 1/3

23. 5/6
- 1/6

24. Circle the smallest: 1/9 1/2 1/4 1/6

25. Circle the largest: 1/8 1/3 1/10 1/4

26. Circle the smallest: 1/4 5/6 3/5

27. Circle the largest: 1/3 6/7 2/5

28. Circle the fraction that has the same value as 1/4: 1/2 2/4 2/8 4/4

29. Circle the fraction that has the same value as 6/10: 5/3 3/6 8/5 3/5 1/6

30. Circle the fraction that has the same value as 1: 1/3 1/4 2/3 5/5 10/11

APPENDIX D
TASKS (REVISED)

Prefatory statement to be read to students:

1. These tests are not timed; you may take all the time you need. You will take one part today, the rest tomorrow.
2. Some questions are very easy. Do not suspect a trick.
3. Others are not as easy. If you don't know, it's okay to say so.
4. I will not be telling you the answers. Nor will I tell you if you are right or wrong. Many questions have more than one correct answer. Often I will ask "Why" or "How do you know?" That doesn't mean it's wrong; it means I want you to explain how you figured it out.
5. I can't write down everything you say and do. So I will ask questions so that you can say, for the tape player, what you are doing.

I. Conservation of number

A. Concrete or manipulative

1. Give one of us 8 chips, the other 10 chips. Extra chips are available. Ask: CAN YOU FIX THESE SO THAT YOU HAVE JUST AS MANY AS I DO?
2. Say: NOW WE HAVE THE SAME NUMBER OF CHIPS. Spread out one row of chips into a long line. Say: NOW WHO HAS MORE CHIPS, YOU OR ME OR ARE THEY THE SAME?

3. Put 20 chips out. Say: CAN YOU DIVIDE THESE CHIPS UP INTO FOUR EQUAL SETS? Spread out two of the sets. Say: ARE THERE MORE HERE OR MORE HERE OR ARE THEY THE SAME?
4. Arrange 12 black chips in 3 by 4 array and 12 blue chips in two 2 by 3 arrays. Say: ARE THERE MORE BLACK CHIPS OR MORE BLUE CHIPS OR ARE THEY THE SAME?

B. Pictorial

1. Present picture of 6 red chips in a row, and 6 blue chips in a row, but with the blue chips spread out in a longer row. Say: ARE THERE MORE BLUE CHIPS OR MORE RED CHIPS OR ARE THEY THE SAME?
2. Present picture of 18 blue chips in a 3 by 6 array, and 18 red chips in two 3 by 4 arrays. Say: ARE THERE MORE BLUE CHIPS OR MORE RED CHIPS OR ARE THEY THE SAME?

II. <u>Seriation</u>

A. Concrete or manipulative

1. Present 10 drinking straws, ranging in length from about 10 to 25 cm. Say: CAN YOU PUT THESE IN ORDER?
2. Present card, with five of the straws attached, arranged in descending order. Present five other (loose) straws of intermediate lengths. Say: SOME OF THESE HAVE ALREADY BEEN PUT IN ORDER. CAN YOU PUT THESE WHERE THEY GO?

B. Pictorial

Present diagram of straws arranged in order by length, the space between them marked by letters a through h. On the right in the picture is another straw. Say: IF YOU WERE TO PUT THIS STRAW IN ORDER WITH THE OTHERS, WHERE WOULD IT GO? WHAT LETTER IS THAT?

III. <u>Classification</u>

A. Concrete or manipulative

1. Present collection of plastic circles and triangles in 2 sizes and 4 colors. Group them by color. Say: I HAVE GROUPED THESE SO THAT THEY GO TOGETHER. IF YOU WERE GOING TO NAME THESE GROUPS, WHAT WOULD YOU CALL THEM?
2. CAN YOU GROUP THEM ANOTHER WAY SO THAT THEY GO TOGETHER? Get them to name or describe the group.
3. ANOTHER WAY?

B. Pictorial

1. Present pictures cut out of magazines (automobiles, food, furniture, men, and a few odd items: a watch, a telephone, Mickey Mouse, a suitcase, a portrait of an eighteenth century gentleman). Ask: CAN YOU GROUP THESE IN SOME WAY SO THAT THEY GO TOGETHER?
2. WHAT SHOULD YOU CALL EACH OF THESE GROUPS?

IV. Class inclusion

A. Concrete or manipulative

1. Present 8 geometric shapes, all red, consisting of 5 circles, 2 squares, and 1 triangle. Say: HERE ARE SOME RED SHAPES. CAN YOU GROUP THESE SO THAT THEY ARE ALIKE IN SOME WAY? WHAT WOULD YOU CALL THIS GROUP? AND THIS GROUP?

2. Say: IF I TOOK ALL THE CIRCLES AWAY, WOULD THERE BE ANY RED SHAPES LEFT?

   IF I TOOK ALL THE RED SHAPES AWAY, WOULD THERE BE ANY CIRCLES LEFT?

3. ARE THERE MORE RED SHAPES OR MORE CIRCLES?

B. Pictorial

1. Present pictures of food (8 of fruit, 4 others: cake, meat, peas, cereal). Ask: CAN YOU THINK OF A NAME TO CALL ALL THESE PICTURES?

   CAN YOU MAKE SOME GROUPS OF FOODS THAT ARE ALIKE?

2. WHAT WOULD YOU CALL THESE GROUPS OF FOOD?

3. IF I TOOK AWAY ALL THE FOOD WOULD THERE BE ANY FRUIT LEFT?

   IF I TOOK AWAY ALL THE FRUIT WOULD THERE BE ANY FOOD LEFT?

4. IS THERE MORE FOOD OR MORE FRUIT?

V. Conservation of distance

A. Concrete or manipulative

1. Present two toy soldiers 6 cm tall. Place them

on the table about 40 cm apart. Ask: ARE THEY CLOSE TOGETHER OR FAR APART?

2. Place a low screen, or barrier, like a ruler, between the two men. As: ARE THEY STILL AS CLOSE TOGETHER (or FAR APART, depending on subject's first response)?
3. Place a cardboard screen higher than the toys (about 15 cm high) between them. Ask: HOW ABOUT NOW? ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?
4. Place a block of wood about 7 cn wide between the toys. ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

B. Pictorial

1. Present drawing of two stick figures about 30 cm apart, with line drawn to represent the horizontal. Ask: ARE THESE TWO PEOPLE CLOSE TOGETHER OR FAR APART?
2. Place short fence made of construction paper between the two figures, flat and perpendicular to the line. Ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?
3. Place taller "fence" made of construction paper (taller than the figures) between the two figures. Ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?
4. Place a wide obstruction of construction paper (about 6 cm wide) between the two figures. Ask: ARE THEY STILL AS CLOSE TOGETHER (FAR APART)?

VI. <u>Conservation of area</u>

A. Concrete or manipulative

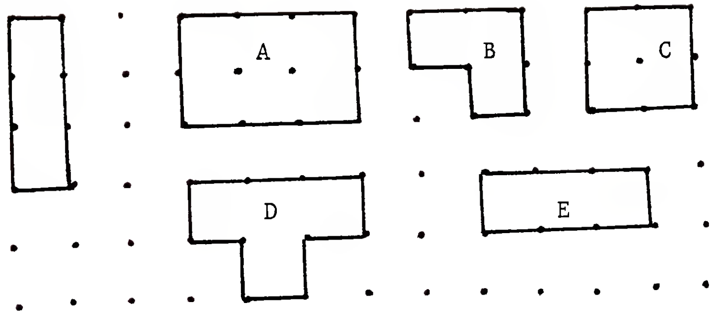
1. a. Present two construction paper "farms." Put a "cow" on each farm. Ask: DO BOTH OF THESE COWS SEEM TO HAVE THE SAME AMOUNT OF GRASS TO EAT?

   b. Put "barn" on each farm. Ask: DOES THIS COW STILL HAVE THE SAME AMOUNT OF GRASS TO EAT AS THIS ONE? If "no," WHICH ONE HAS MORE? WHY?

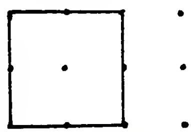
   c. Put additional barn on each farm. On one farm, put the barn right next to the first one; on the other farm, put the barn some distance away. Ask: NOW DO BOTH COWS HAVE THE SAME AMOUNT OF GRASS TO EAT? If "no," WHICH ONE HAS MORE? WHY?

   d. Continue adding barns, juxtaposed on the one farm and spread out on the other, continuing to ask each time: NOW DO BOTH COWS HAVE THE SAME AMOUNT OF GRASS TO EAT (AS EACH OTHER)?

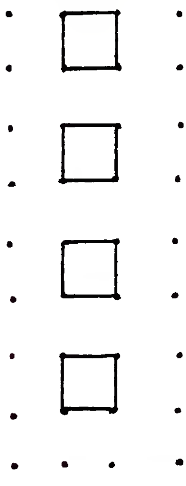
2. a. Present geoboard and rubber bands. Make a standard figure with a blue rubber band (enclosing 3 units of area). Then make other shapes (all unions of rectangles, no triangles) with other rubber bands. Ask: ARE THERE ANY OF THESE SHAPES THAT HAVE JUST AS MUCH ROOM INSIDE AS THIS BLUE ONE DOES? If one is named, ARE THERE ANY OTHERS? etc.



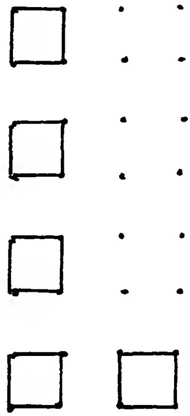
2. b. Make standard figure (4 units of area) with a blue rubber band. Ask: CAN YOU USE THE RUBBER BANDS TO MAKE A SHAPE THAT HAS JUST AS MUCH ROOM INSIDE AS THIS SHAPE?



c. Use four rubber bands to enclose 4 (individual) square units in a column. Ask: CAN YOU USE THE RUBBER BANDS TO MAKE ONE SHAPE THAT HAS JUST AS MUCH ROOM INSIDE AS THESE FOUR SHAPES?

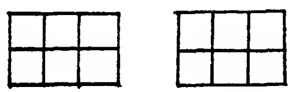


2. d. Add fifth unit of area to the four in c, but not in the column. Ask: CAN YOU MAKE ONE SHAPE THAT HAS JUST AS MUCH ROOM INSIDE AS THESE FIVE SHAPES?

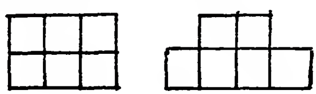


B. Pictorial

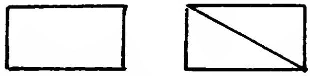
1. a. Present two rectangles made of construction paper squares. Ask: ARE THESE TWO RECTANGLES SAME SIZE? DO THEY HAVE THE SAME AMOUNT OF SPACE INSIDE? Or, DO THEY TAKE UP THE SAME AMOUNT OF ROOM?



b. Rearrange one of the squares in one of the rectangles to make a different shape. Ask: WHAT ABOUT NOW? DO THEY TAKE UP THE SAME AMOUNT OF SPACE (ROOM)?



B. 2. a. Present two construction paper rectangles, one of which is cut along the diagonal to form two triangles.



Ask: DO THESE TWO RECTANGLES TAKE UP THE SAME AMOUNT OF SPACE (ROOM)?

b. Rearrange one of the rectangles into an isosceles triangle.

Ask: DO THESE TWO SHAPES TAKE UP THE SAME AMOUNT OF SPACE (ROOM)?

APPENDIX E
CONCRETE FRACTIONS TEST
(REVISED)

Prefatory statement to be read to students

1. This is not a timed test.
2. You may not know all of the answers. If you don't know one, it's okay to say, "I don't know." It's okay to guess if you want to.
3. I will be showing you some things and I will be asking you some questions. Sometimes I will give you an example first.
4. Please speak up!
5. Often I will ask you "Why?" of "How do you know?" That means I want you to explain how you decided.

The investigator will do:	will say:
A. Concept of fractions (Discrete model)	
(Example) Arrange 4 blue chips and 3 white chips.	This is an example. Here there are 4 blue chips out of 7. We say that 4/7 of the chips are blue.
1. Arrange 3 blue chips and 5 white chips in a row.	1. What fraction of the chips are blue?
2. Make a 2 by 3 array, 2 blue, 4 white chips.	2. a. What fraction of the chips are blue now? b. Can you say in another what fraction of the

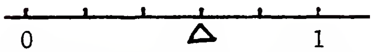
	chips are blue?
3. Rearrange array; put 3 blue, but alternating, not next to each other, and 3 white chips.	3. a. Now what fraction of the chips are blue? b. Can you say in another way what fraction of the chips are blue?
4. Make an array of 10 blue chips. Give the subject a string knotted together at the ends.	4. Can you put this string around 3/10 of the chips?
5. Make available chips of both colors, also string.	5. Can you use the chips to show what is meant by the fraction 2/5? (If necessary) Show me which part is 2/5.
6. Make chips and string available.	6. Can you show what is meant by 5/5?
7. Make chips and string available.	7. Can you show what is meant by 0/5?

B. <u>Concept of fraction</u>

<u>(Number line model)</u>

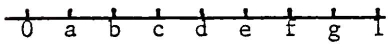
Example: Show string mounted on cardboard, with fifths marked in black magic marker, triangle pointing to the third mark.

This is an example. Look at the distance from zero to the triangle. That distance is 3/5.



1. Show different string mounted on cardboard, marked off in eighths, with letters a through g at 1/8 through 7/8.

1. Which letter is at one half?



2. Show same model as in 1.

2. What fraction is represented by the letter a?

3. Show same model as in 1 and 2.

3. Which letter is at 3/4?

4. Show string mounted on cardboard with only the zero and one marked.

4. Put your pencil point about where 1/4 should be on this number line.

5. Show same model as in 4.

5. Put your pencil point about where 3/6 should be on this number line.

C. <u>Concept of fraction (Area model)</u>

Example: Arrange small yellow and red rectangles into one rectangle with 3 yellow and 1 red.

This is an example. I'm making one large rectangle from these small rectangles. The fraction that is red is one fourth.

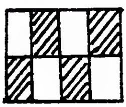
1. Arrange rectangle with 3 red and 2 yellow.

1. What fraction of the large rectangle is red?

2. Make rectangle with 3 red and 3 yellow.

2. a. Now what fraction of the large rectangle is red?
   b. Can you say another way what fraction is red?

3. Arrange 4 red and 4 yellow in an alternating fashion.



3. a. What fraction of this rectangle is red?
   b. Can you say another way what fraction is red?

4. Put out several red and yellow rectangles in no order.	4. Using as many of these as you wish, arrange them to show what is meant by the fraction 1/2.
5. Make red and yellow rectangles available.	5. Now arrange some of them to show what is meant by the fraction 2/7.
6. Make red and yellow rectangles available.	6. Can you show what is meant by 3/3?

D. Equivalent fractions

Example: Make row of 10 chips. Then put the string around 5 of them.	This is an example. I'm putting the string around one half of the chips.
1. Make array of 12 chips, 2 by 6.	1. Put the string around 9/12 of the chips.
2. Mess up chips and arrange them into 3 by 4 array.	2. Now put the string around 1/3 of the chips.
3. Put 8 chips out, not in neat array.	3. Can you put the string around 1/4 of the chips?
4. Put out 12 chips, not arranged.	4. Put the string around 3/4 of the chips.
5. Put out 15 chips: 10 black, 5 blue.	5. a. What fraction of these chips are black? b. Can you say another way what fraction of the chips are black?

6. Put out 15 black chips.	6. What fraction of these chips are black?

E. <u>Comparing fractions</u>

Example: Put out 6 chips, 3 pink and 3 blue.	This is an example. I want to see which is bigger, 1/3 or 1/2. I am making one half of these chips blue.
Put out six more chips; show obvious sets of 2, with 2 blue, 4 pink.	I am making one third of these chips blue.
	I can see now that 1/2 is bigger than one third.
1. Make chips available.	1. Can you find a way to decide with fraction is bigger, 5/8 or 3/4?

(If successful, continue)

2. Make chips available.	2. Can you show which is bigger, 5/12 or 1/3?

APPENDIX F
WRITTEN FRACTIONS TEST
(REVISED)

Prefatory statement to be read to students

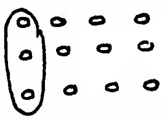
1. You may use scratch paper if you need it.
2. There is no time limit.
3. You may not know all of the answers. If you don't know one, you may either skip it and go on, or you may guess at it if you want to.
4. If you don't know a word in the directions, please ask. If you don't understand the question, please ask.
5. The questions are on cards so that you only have to look at one question at a time. When you finish one, turn it over and go on to the next one.

A. Concept of fraction (Discrete model)

1. Circle the fraction this picture shows:

   3/4 3/7 1/2 5/3

2. Circle the fraction this picture shows:



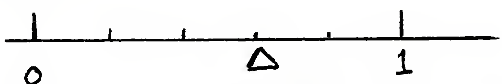
   3/9 1/2 1/4 1/3

3. Circle 5/9 of the chips:

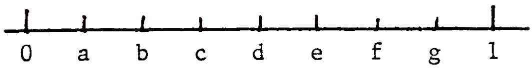
4. Circle 1/2 of the chips:

B. Concept of fraction (Number line model)

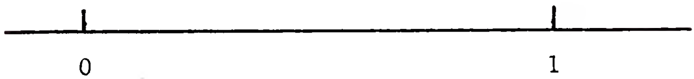
1. Circle the answer. Which fraction names the distance from O to Δ ? 3 1/3 3/5 3/4 1/2



2. Which letter is at 5/8? ______



3. On the number line, put a dot about where 1/3 should be.

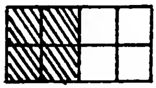


4. On the line, put a dot about where 4/8 should be.

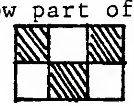
C. Concept of fraction (Area model)

1. What fraction of the rectangle is red? ________

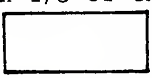
2. The fraction that names the green part of the rectangle is_________.



3. The fraction that names the yellow part of the rectangle is__________.



4. Color in 1/3 of this rectangle.



5. Color in 2/5 of this rectangle.

D. Equivalent fractions

1. Circle the fraction that has the same value as 1/2:

   1/1 2/3 3/4 3/6 4/10

2. Circle the fraction that has the same value as 6/10:

   5/3 3/6 8/5 3/5 1/6

3. Circle the fraction that has the same value as 1:

   1/3 1/4 2/3 5/5 10/11

4. Circle the fraction that has the same value as 1/4:

   1/2 2/4 2/8 4/4

E. Comparing fractions

1. Circle the smallest: 1/9 1/2 1/4 1/6
2. Circle the largest: 1/8 1/3 1/10 1/4
3. Circle the smallest: 1/3 3/4 1/2
4. Circle the largest: 1/3 3/5 2/7
5. Put these fractions in order, from the smallest to the largest: 3/4 1/16 1/2 3/8

F. Addition and subtraction of fractions

1. 2/5 + 1/5 =

2. 3/7 + 2/7

3. 1/2 + 1/2

4. 3/7 - 1/7 =

5. 7/8 - 5/8

6. 1/3 - 1/3

APPENDIX G
TASK RESULTS

Table G-1
I. Conservation of Number

Subtask		Pass	Percentage of Students Questionable	Skip	Skip(e)
A. Concrete	1	84	16		
	2	100			
	3a	52	32		
	b	64	4	20	8
	4	64	16		
B. Pictorial	1	88	8		
	2	76	12		

Skip--Skipped because student missed prerequisite question
Skip(e)--Skipped through investigator error

Table G-2
II. Seriation

Subtask		Percentage of Students Pass	Questionable
A. Concrete	1	96	
	2	56	8
B. Pictorial		88	

Table G-3
III. Classification

Subtask		Percentage of Students: Pass	Questionable	Skip
A. Concrete	1	88	8	
	2	52	20	
	3	20	20	4
B. Pictorial	1	60	32	
	2	40	48	

Skip--Skipped because student missed prerequisite question

Table G-4
IV. Class Inclusion

Subtask		Percentage of Students: Pass	Questionable
A. Concrete	1	96	4
	2a	92	4
	b	84	4
	3*	56	4
B. Pictorial	1	96	4
	2	88	12
	3a	80	
	b	76	12
	4*	8	4

*Questions A3 and B4 are the determining questions for class inclusion

Table G-5
V. Conservation of Distance

Subtask		Percentage of Students Pass	Questionable
A. Concrete	1*	100	
	2	88	
	3	80	8
	4	72	8
B. Pictorial	1*	100	
	2	88	
	3	88	
	4	64	8

*Questions A1 and B1 set frame of reference only

Table G-6
VI. Conservation of Area

Subtask		Percentage of Students Pass	Questionable
A. Concrete	1	52	4
	2a	0	4
	b	0	8
	c	4	8
	d	4	4
B. Pictorial	1	16	4
	2	4	16

APPENDIX H
CONCRETE FRACTIONS TEST RESULTS

Table H-1
A. Concept of Fraction (Discrete Model)

Item	Percentage of Students Pass	Questionable	Skip
1	32	12	
2a	36	12	4
b*	12	4	12
3a	32	16	4
b*	12		8
4	64	8	
5	52	4	4
6	28		4
7	32	4	4

Skip--Skipped because student failed prerequisite question

*Questions 2b and 3b are not crucial to this section

Table H-2
B. Concept of Fraction (Number Line Model)

Item	Percentage of Students Pass	Questionable	Skip	Skip(e)
1	40			4
2	8	4	4	
3	12	4		
4	12	4		
5	0	8	4	

Skip--Skipped because student failed prerequisite question

Skip(e)--Skipped through investigator error

Table H-3
C. Concept of Fraction (Area Model)

Item	Pass	Percentage of Students Questionable	Skip	Skip(e)
1	76			
2a	52	12		
b*	20		8	
3a	56	12		
b*	24		4	
4	68			
5	68	4	4	
6	16	8	4	4

Skip--Skipped because student failed prerequisite question

Skip(e)--Skipped through investigator error

*Questions 2b and 3b were not crucial to this section

Table H-4
D. Equivalent Fractions

Item	Pass	Percentage of Students Questionable	Skip
1	80		4
2	8	12	4
3	4	4	4
4	4	8	4
5	8	16	4
6	12	20	4

Skip--Skipped because student failed prerequisite question

Table H-5
E. Comparing Fractions

No student was successful on E.

APPENDIX I
WRITTEN FRACTIONS TEST RESULTS

Table I-1
A. Concept of Fraction (Discrete Model)

Item	Percentage of Students Pass	Questionable
1	44	
2	12	
3	84	
4	68	4

Table I-2
B. Concept of Fraction (Number Line Model)

Item	Percentage of Students Passing
1	4
2	56
3	28
4	12

Table I-3
C. Concept of Fraction (Area Model)

Item	Percentage of Students Pass	Questionable
1	48	
2	68	
3	60	
4	36	4
5	4	8

Table I-4
D. Equivalent Fractions

Item	Percentage of Students Passing
1	24
2	44
3	64
4	44

Table I-5
E. Comparing Fractions

Item	Percentage of Students Passing
1	16
2	20
3	16
4	24
5	0

Table I-6
F. Adding and Subtracting Fractions

	Item	Percentage of Students Passing
Adding	1	44
	2	48
	3	48
Subtracting	4	48
	5	52
	6	32

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Roberta Edna Lea was born August 9, 1938, in Big Spring, Texas. She moved to Florida in her youth, married Carroll Dees in 1957, and raised five children.

She received the Bachelor of Arts degree in mathematics from Duke University in 1960 and the Master of Science degree in mathematics from Florida State University in 1967.

She has spent fifteen years in public education in Florida, grades seven through twelve, junior college, and adult school. In addition to various mathematics and science courses, she has taught English, reading, health, Latin, and forensics.

Her primary professional interests have been in teaching disadvantaged students (she was director of Special Services for Disadvantaged and Handicapped at Lake City Community College for two years), individualized programs (she implemented an Individualized Manpower Training System in Columbia County schools), and the laboratory method of teaching mathematics. She is currently employed as a research associate in mathematics education at the University of Chicago.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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